2017.2 Question 8

The line through A perpendicular to BC is

$$l_1 : \mathbf{r} = \mathbf{a} + \lambda \mathbf{u}, \lambda \in \mathbb{R}.$$

The line through B perpendicular to CA is

$$l_2 : \mathbf{r} = \mathbf{b} + \mu \mathbf{v}, \mu \in \mathbb{R}.$$

Since P is the intersection of l_1 and l_2, we must have

$$\mathbf{a} + \lambda \mathbf{u} = \mathbf{b} + \mu \mathbf{v},$$

and hence solving for $\mathbf{v}$ we have

$$\mathbf{v} = \frac{1}{\mu} \left(\mathbf{a} + \lambda \mathbf{u} - \mathbf{b} \right).$$

Since $\mathbf{v}$ is perpendicular to CA, we must have $\mathbf{v} \cdot (\mathbf{a} - \mathbf{c}) = 0$, and hence

$$\begin{aligned} & \frac{1}{\mu} \left(\mathbf{a} + \lambda \mathbf{u} - \mathbf{b} \right) \cdot (\mathbf{a} - \mathbf{c}) = 0 \\ \iff & (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) + \lambda \mathbf{u} \cdot (\mathbf{a} - \mathbf{c}) = 0 \\ \iff & \lambda = -\frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})}. \end{aligned}$$

Hence, the position vector of P, $\mathbf{p}$, must satisfy that

$$\mathbf{p} = \mathbf{a} + \lambda \mathbf{u} = \mathbf{a} - \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})} \mathbf{u}.$$

CP is perpendicular to AB if and only if $(\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) = 0$. We notice

$$\begin{aligned} (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) &= (\mathbf{a} + \lambda \mathbf{u} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \left(\mathbf{b} - \mathbf{a} \right). \end{aligned}$$

Since $\mathbf{u}$ is perpendicular to BC, we must have $\mathbf{u} \cdot (\mathbf{c} - \mathbf{b})$, and hence $\mathbf{u} \cdot \mathbf{c} = \mathbf{u} \cdot \mathbf{b}$. Hence,

$$\begin{aligned} (\mathbf{p} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \cdot (\mathbf{b} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + \lambda \mathbf{u} \cdot (\mathbf{c} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) - \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c})}{\mathbf{u} \cdot (\mathbf{a} - \mathbf{c})} \mathbf{u} \cdot (\mathbf{c} - \mathbf{a}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{c}) \\ &= (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) - (\mathbf{a} - \mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= 0, \end{aligned}$$

and hence CP is perpendicular to AB.