2017.2 Question 7

1. Since ln is an increasing function, for $0 < x < 1$, we have $\ln x < 0$, and

$$\begin{aligned} f(x) > x &\iff \ln f(x) > \ln x \\ &\iff \ln x^x > \ln x \\ &\iff x \ln x > \ln x \\ &\iff x < 1, \end{aligned}$$

which is true since $0 < x < 1$.

Notice that

$$\begin{aligned} x < g(x) < f(x) &\iff \ln x < \ln x^{f(x)} < \ln x^x \\ &\iff \ln x < x^x \ln x < x \ln x \\ &\iff 1 > x^x > x. \end{aligned}$$

The right inequality is shown by the previous part. For the left inequality, we have

$$\begin{aligned} 1 > x^x &\iff \ln 1 > x \ln x \\ &\iff 0 > x \ln x \end{aligned}$$

must be true, since $0 < x < 1$ and $\ln x < 0$.

Hence, we have $x < g(x) < f(x)$ for $0 < x < 1$.

When $x > 1$, we claim that $x < f(x) < g(x)$.

2. Notice that

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} x^x \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \exp(x \ln x) \\ &= \exp(x \ln x) \cdot \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \\ &= \exp(x \ln x) \cdot (\ln x + 1) \\ &= f(x) \cdot (\ln x + 1). \end{aligned}$$

$f'(x) = 0$ if and only if $\ln x + 1 = 0$, which holds if and only if $x = \frac{1}{e}$.

3. We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \exp(x \ln x) = \exp(0) = 1,$$

and hence

$$\lim_{x \to 0} g(x) = 0.$$

4. Let $h(x) = \frac{1}{x} + \ln x$. We have

$$h'(x) = -\frac{1}{x^2} + \frac{1}{x} = \frac{x-1}{x^2}.$$

When $0 < x < 1$, $h'(x) < 0$, and when $1 < x$, $h'(x) > 0$. Hence, h takes a minimum when $x = 1$, and $h(1) = \frac{1}{1} + \ln 1 = 1$.

This shows precisely that

$$\frac{1}{x} + \ln x \geq 1$$

for $x > 0$.

Notice that

$$\begin{aligned}
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} x^{f(x)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \exp(f(x)\ln x) \\
&= \exp(f(x)\ln x) \cdot \left(\frac{1}{x} \cdot f(x) + f'(x)\ln x\right) \\
&= g(x) \cdot \left(\frac{1}{x} \cdot f(x) + f(x) \cdot (\ln x + 1) \cdot \ln x\right) \\
&= f(x)g(x) \cdot \left(\frac{1}{x} + \ln x(\ln x + 1)\right) \\
&\geq f(x)g(x) \cdot \left(1 + (\ln x)^2\right) \\
&> 0,
\end{aligned}$$

since $f(x), g(x) > 0$ for $x > 0$ (since they are both exponentials), and $1 + (\ln x)^2 \geq 1 > 0$ as well.

The graphs of the functions look as follows.