2017.2 Question 6

1. We first look at the base case where $n = 1$. $S_1 = \frac{1}{1} = 1$, and $2 \cdot \sqrt{1} - 1 = 1$, so
$$S_1 \leq 2 \cdot \sqrt{1} - 1$$
holds, and the original statement holds for when $n = 1$.

   Assume this holds for some $n = k \in \mathbb{N}$, i.e., $S_k \leq 2\sqrt{k} - 1$. We have
$$\begin{aligned} S_{k+1} &= \sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} \\ &= \sum_{r=1}^{k} \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{k+1}} \\ &= S_k + \frac{1}{\sqrt{k+1}} \\ &\leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} - 1. \end{aligned}$$
   We would like to show
$$2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1}.$$
   Notice that
$$\begin{aligned} 2\sqrt{k} + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} &\iff 2\sqrt{k(k+1)} + 1 \leq 2(k+1) \\ &\iff 2\sqrt{k(k+1)} \leq 2k+1 \\ &\iff 4k(k+1) \leq (2k+1)^2 \\ &\iff 4k^2 + 4k \leq 4k^2 + 4k + 1 \\ &\iff 0 \leq 1, \end{aligned}$$
   which is true.

   Hence,
$$S_{k+1} \leq 2\sqrt{k} + \frac{1}{\sqrt{k+1}} - 1 \leq 2\sqrt{k+1} - 1,$$
   which is precisely the statement for $n = k + 1$.

   The original statement holds for the base case where $n = 0$, and assuming it holds for some $n = k \in \mathbb{N}$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, the original statement holds for all $n \in \mathbb{N}$.

2. For $k \geq 0$, we notice
$$\begin{aligned} (4k+1)\sqrt{k+1} > (4k+3)\sqrt{k} &\iff (4k+1)^2(k+1) > (4k+3)^2 k \\ &\iff (16k^2 + 8k + 1)(k+1) > (16k^2 + 24k + 9)k \\ &\iff 16k^3 + 8k^2 + k + 16k^2 + 8k + 1 > 16k^3 + 24k^2 + 9k \\ &\iff 1 > 0, \end{aligned}$$
   which is true.

   We claim that $C = \frac{3}{2}$ is the smallest number C which makes this true. We first show that $C = \frac{3}{2}$ makes the statement true by induction. For the base case where $n = 1$, $S_1 = 1$, and
$$2\sqrt{1} + \frac{1}{2\sqrt{1}} - \frac{3}{2} = \frac{5}{2} - \frac{3}{2} = 1,$$
   and so this statement holds for $n = 1$.

Now, assume that this statement holds for some $n = k \in \mathbb{N}$, i.e.

$$S_k \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} - C.$$

We have

$$\begin{aligned} S_{k+1} &= S_k + \frac{1}{\sqrt{k+1}} \\ &\geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - C. \end{aligned}$$

We would like to show that

$$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}}.$$

Notice that

$$\begin{aligned} & 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} \\ \iff & 2\sqrt{k} + \frac{1}{2\sqrt{k}} \geq 2\sqrt{k+1} - \frac{1}{2\sqrt{k+1}} \\ \iff & \frac{4k+1}{2\sqrt{k}} \geq \frac{4(k+1)-1}{2\sqrt{k+1}} \\ \iff & (4k+1)\sqrt{k+1} \geq (4k+3)\sqrt{k}, \end{aligned}$$

which is implied by the proven inequality, and hence

$$S_{k+1} \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - C \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - C,$$

which precisely proves the statement for $n = k + 1$.

The claimed statement holds for the base case where $n = 1$, and given it holds for some $n = k \in \mathbb{N}$, it holds for $n = k + 1$. Hence, the statement holds for all $n \in \mathbb{N}$ when $C = \frac{3}{2}$.

If $C < \frac{3}{2}$, we have for $n = 1$

$$2\sqrt{1} + \frac{1}{2\sqrt{1}} - C > \frac{5}{2} - \frac{3}{2} = 1,$$

but

$$S_1 = 1,$$

so the statement does not hold for when $n = 1$.

Hence, the smallest number C for the statement to be true is $C = \frac{3}{2}$.