2017.2 Question 5

1. By taking derivatives with respect to t, we have
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2at,$$
and
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2a,$$
hence
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2at} = \frac{1}{t}.$$
The gradient of the normal will hence be $-t$, and hence the normal through $P(ap^2, 2ap)$ will be
$$y - 2ap = -p(x - ap^2).$$
The point $N(an^2, 2an)$ is also on this line, and hence
$$2a(n-p) = -ap(n-p)(n+p).$$
Since $n \neq p$, we must have
$$2 = -p(n+p).$$
Given $p \neq 0$, we have
$$n + p = -\frac{2}{p},$$
and hence
$$n = -p - \frac{2}{p} = -\left(p + \frac{2}{p}\right).$$

2. The distance between $P(ap^2, 2ap)$ and $N(an^2, 2an)$ is given by
$$\begin{aligned}|PN|^2 &= (2ap - 2an)^2 + \left(ap^2 - an^2\right)^2 \\ &= a^2\left[4(p-n)^2 + (p-n)^2(p+n)^2\right] \\ &= a^2(p-n)^2\left[4 + 4\left(-\frac{2}{p}\right)^2\right] \\ &= a^2\left[p + \left(p + \frac{2}{p}\right)\right]^2 \cdot 4\left(\frac{p^2+1}{p^2}\right) \\ &= 4a^2 \cdot 4 \cdot \frac{(p^2+1)^2}{p^2} \cdot \frac{p^2+1}{p^2} \\ &= 16a^2\frac{(p^2+1)^3}{p^4}.\end{aligned}$$
Let $f(p) = \frac{(p^2+1)^3}{p^4}$. By differentiation,
$$\begin{aligned}f'(p) &= \frac{3 \cdot 2p \cdot (p^2+1)^2 \cdot p^4 - (p^2+1)^3 \cdot 4 \cdot p^3}{p^8} \\ &= \frac{2(p^2+1)^2p^3}{p^8}\left[3p^2 - 2(p^2+1)\right] \\ &= \frac{2(p^2+1)^2}{p^5}\left(p^2 - 2\right).\end{aligned}$$
This means that $f'(p) = 0$ precisely when $p^2 - 2 = 0$, i.e. $p = \pm\sqrt{2}$.
When $0 < p < \sqrt{2}$, $f'(p) < 0$, and when $\sqrt{2} < p$, $f'(p) > 0$.
When $p < -\sqrt{2}$, $f'(p) < 0$, and when $-\sqrt{2} < p < 0$, $f'(p) > 0$.
This means that when $p^2 - 2 = 0$ (i.e. $p = \pm\sqrt{2}$), $f(p)$ has a minimum.
Since $|PN|^2 = \frac{16}{a^2}f(p)$ is a positive multiple of $f(p)$, we must have that $|PN|^2$ is minimised when $p^2 = 2$.

3. Since $Q(aq^2, 2aq)$ is on the circle with diameter PN, we must have that QP and QN are perpendicular.

   The gradient of QP is given by

$$m_{QP} = \frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2(q-p)}{(q+p)(q-p)} = \frac{2}{q+p},$$

   and the gradient of QN is given by

$$m_{QN} = \frac{2aq - 2an}{aq^2 - an^2} = \frac{2(q-n)}{(q+n)(q-n)} = \frac{2}{q+n}.$$

   Since QP and QN are perpendicular, we must have

$$\begin{aligned} m_{QP} \cdot m_{QN} = -1 &\iff \frac{2}{q+p} \cdot \frac{2}{q+n} = -1 \\ &\iff -4 = (q+p)(q+n) \\ &\iff q^2 + (p+n)q + pn = -4 \\ &\iff q^2 - \frac{2}{p} \cdot q - p^2 - 2 = -4 \\ &\iff p^2 - q^2 + \frac{2q}{p} = 2, \end{aligned}$$

   as desired.

   When $|PN|$ is a minimum, we have $p = \pm\sqrt{2}$, and hence

$$2 - q^2 \pm \sqrt{2}q = 2,$$

   which gives

$$q(q \mp \sqrt{2}) = 0.$$

   Hence, $q = 0$, or $q = \pm\sqrt{2}$ (which means $p = q$, which cannot be the case). When $q = 0$, $Q(0, 0)$ is at the origin, as desired.