2017.2 Question 4

1. If $f(x) = 1$, this gives
$$\left(\int_a^b g(x)\,\mathrm{d}x\right)^2 \leq (b-a)\left(\int_a^b g(x)^2\,\mathrm{d}x\right).$$
Let $g(x) = e^x$, $a = 0$ and $b = t$, and we have
$$\text{LHS} = \left(\int_0^t e^x\,\mathrm{d}x\right)^2 = \left(e^t - 1\right)^2,$$
and
$$\text{RHS} = t\int_0^t e^{2x}\,\mathrm{d}x = \frac{t}{2}\left(e^{2t} - 1\right) = \frac{t}{2}\left(e^t - 1\right)\left(e^t + 1\right).$$
Since LHS $\leq$ RHS, we have
$$\left(e^t - 1\right)^2 \leq \frac{t}{2}\left(e^t - 1\right)\left(e^t + 1\right),$$
and hence
$$\frac{e^t - 1}{e^t + 1} \leq \frac{t}{2},$$
since $e^t + 1 > 0$.

2. If $f(x) = x$, and $a = 0, b = 1$, the Schwarz inequality gives
$$\left(\int_0^1 xg(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 x^2\,\mathrm{d}x \int_0^1 g(x)^2\,\mathrm{d}x.$$
Since
$$\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3},$$
we therefore have
$$3\left(\int_0^1 xg(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 g(x)^2\,\mathrm{d}x.$$
Consider $g(x) = \exp\left(-\frac{1}{4}x^2\right)$. Notice that
$$\begin{aligned}\int_0^1 xg(x)\,\mathrm{d}x &= \int_0^1 x\exp\left(-\frac{1}{4}x^2\right)\mathrm{d}x \\ &= -2\left[\exp\left(-\frac{1}{4}x^2\right)\right]_0^1 \\ &= -2\left[\exp\left(-\frac{1}{4}\right) - \exp(0)\right] \\ &= 2\left(1 - \exp\left(-\frac{1}{4}\right)\right),\end{aligned}$$
and hence
$$3\cdot\left[2\left(1 - \exp\left(-\frac{1}{4}\right)\right)\right]^2 \leq \int_0^1 \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x,$$
which is equivalent to
$$\int_0^1 \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x \geq 12\left(1 - \exp\left(-\frac{1}{4}\right)\right)^2,$$
as desired.

3. For the right-half of the inequality, let $f(x) = 1$, and let the bounds be $a = 0, b = \frac{1}{2}\pi$, we have

$$\left(\int_0^{\frac{\pi}{2}} g(x)\,\mathrm{d}x\right)^2 \leq \frac{\pi}{2}\int_0^{\frac{\pi}{2}} g(x)^2\,\mathrm{d}x.$$

Let $g(x) = \sqrt{\sin x}$, and hence

$$\left(\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x\right)^2 \leq \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}x = \frac{\pi}{2}\left[-\cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

Since the integrand $\sqrt{\sin x} \geq 0$ for all $x \in \left[0, \frac{\pi}{2}\right]$, the integral over this interval must be non-negative, and hence

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x \leq \sqrt{\frac{\pi}{2}}.$$

For the left-half of the inequality, consider $g(x) = \sqrt[4]{\sin x}$, and $f(x) = \cos x$ (with the same bounds, $a = 0, b = \frac{1}{2}\pi$). We notice that

$$\begin{aligned}\int_a^b f(x)g(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} \cos x \sqrt[4]{\sin x}\,\mathrm{d}x \\ &= \frac{4}{5}\left[(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}} \\ &= \frac{4}{5}\left[1^{\frac{5}{4}} - 0^{\frac{5}{4}}\right] \\ &= \frac{4}{5},\end{aligned}$$

and that

$$\begin{aligned}\int_a^b f(x)^2\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} \cos^2 x\,\mathrm{d}x \\ &= \int_0^{\frac{1}{2}\pi} \frac{1+\cos 2x}{2}\,\mathrm{d}x \\ &= \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi} \\ &= \left[\frac{1}{2}\cdot\frac{1}{2}\pi + \frac{1}{4}\sin\pi\right] - \left[\frac{1}{2}\cdot 0 + \frac{1}{4}\sin 0\right] \\ &= \frac{1}{4}\pi.\end{aligned}$$

Hence, by the Schwarz inequality, we have

$$\frac{16}{25} \leq \frac{1}{4}\pi \cdot \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x,$$

and hence

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,\mathrm{d}x \geq \frac{64}{25\pi}.$$

Combining both sides of the equality, we hence have

$$\frac{64}{25\pi} \leq \int_0^{\frac{\pi}{2}} \sqrt{\sin x} \leq \sqrt{\frac{\pi}{2}},$$

as desired.