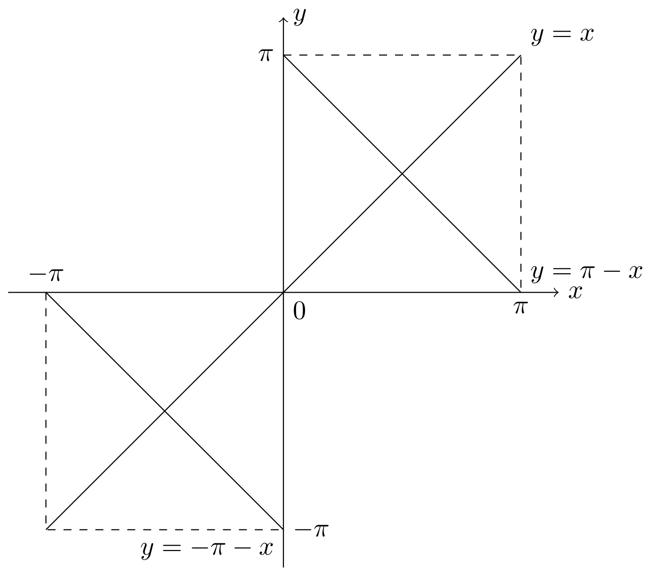
2017.2 Question 3

1. Since $\sin y = \sin x$, we must have
$$y = x + 2k\pi$$
where $k \in \mathbb{Z}$, or
$$y = (2k+1)\pi - x$$
where $k \in \mathbb{Z}$.
For the first case, since $x \in [-\pi, \pi]$ and $y \in [-\pi, \pi]$, we must have simply $x = y$.
For the second case, within this range, we can have $y = \pi - x$, and $y = -\pi - x$.
Hence, the sketch looks as follows.



2. Differentiating with respect to x, we have
$$\cos y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \cos x.$$
Since $\sin y = \frac{1}{2} \sin x$, $\cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1 - \frac{1}{4}\sin^2 x}$. Since $0 \leq y \leq \frac{1}{2}\pi$, $\cos y > 0$, and hence $\cos y = \frac{1}{2}\sqrt{4 - \sin^2 x}$. Hence,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2}\cos x}{\frac{1}{2}\sqrt{4 - \sin^2 x}} = \frac{\cos x}{\sqrt{4 - \sin^2 x}}.$$
Differentiating this again gives us
$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{(-\sin x)\sqrt{4 - \sin^2 x} - \frac{1}{2} \cdot (-2\sin x) \cdot \cos x \cdot \frac{1}{\sqrt{4 - \sin^2 x}} \cdot \cos x}{4 - \sin^2 x} \\ &= \frac{(-\sin x)(4 - \sin^2 x) + \sin x \cos^2 x}{(4 - \sin^2 x)^{\frac{3}{2}}} \\ &= \frac{-4\sin x + \sin^3 x + \sin x(1 - \sin^2 x)}{(4 - \sin^2 x)^{\frac{3}{2}}} \\ &= -\frac{3\sin x}{(4 - \sin^2 x)^{\frac{3}{2}}},\end{aligned}$$
as desired.

Within this range of x and y, we have

$$y = \arcsin\left(\frac{1}{2}\sin x\right),$$

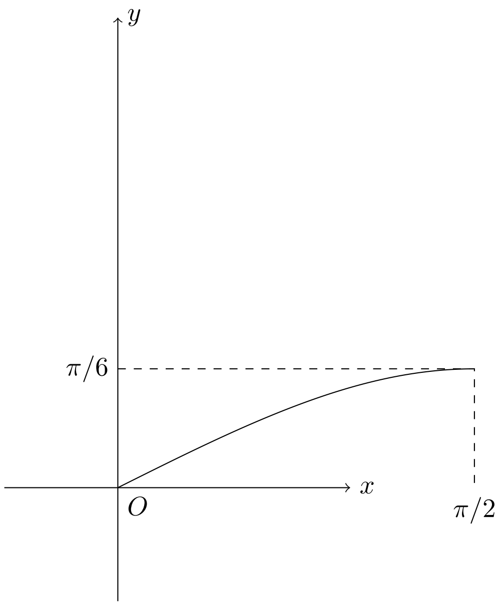
and hence this is a function, and each x corresponds to a unique y.
At $x = 0$,

$$y = 0, y' = \frac{\cos 0}{\sqrt{4 - \sin^2 0}} = \frac{1}{2},$$

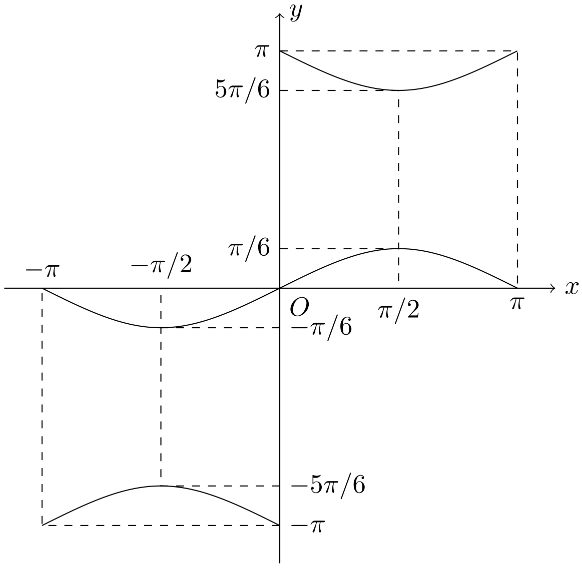
and at $x = \frac{\pi}{2}$,

$$y = \frac{\pi}{6}, y' = -\frac{\cos \frac{\pi}{2}}{\sqrt{4 - \sin^2 \frac{\pi}{2}}} = 0.$$

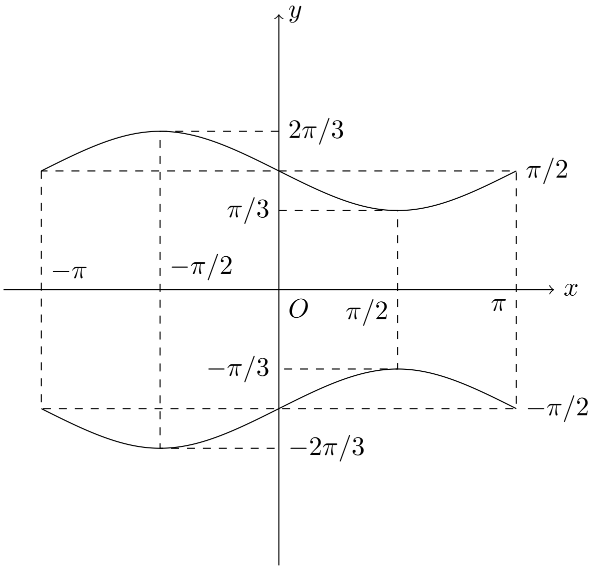
Since $y'' = -\frac{3\sin x}{(4-\sin^2 x)^{\frac{3}{2}}} < 0$ for $x \in \left[0, \frac{\pi}{2}\right]$, this function is concave, and hence the graph looks as follows.



Hence, for $(x, y) \in [-\pi, \pi]^2$, the graph looks as follows, by symmetry.



3. The graph is as follows.



y
$2\pi/3$
$\pi/2$
$\pi/3$
$-\pi$
$-\pi/2$
x
O
$\pi/2$
π
$-\pi/3$
$-\pi/2$
$-2\pi/3$