2017.2 Question 2

We have

$$\begin{aligned} x_{n+2} &= \frac{ax_{n+1}-1}{x_{n+1}+b} \\ &= \frac{a\cdot\frac{ax_n-1}{x_n+b}-1}{\frac{ax_n-1}{x_n+b}+b} \\ &= \frac{a(ax_n-1)-(x_n+b)}{(ax_n-1)+b(x_n+b)} \\ &= \frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}. \end{aligned}$$

1. If the sequence is periodic with period 2, then for all integers $n \geq 0$, we have

$$\begin{aligned} x_{n+2}=x_n &\iff x_n\left[(a+b)x_n+(b^2-1)\right]=(a^2-1)x_n-(a+b) \\ &\iff (a+b)x_n^2-(a+b)(a-b)x_n+(a+b)=0 \\ &\iff (a+b)(x_n^2-(a-b)x_n+1)=0. \end{aligned}$$

   We also have

$$\begin{aligned} x_{n+1}=x_n &\iff x_n(x_n+b)=ax_n-1 \\ &\iff x_n^2-(a-b)x_b+1=0, \end{aligned}$$

   and this means that for some $n=k\geq 0$, we must have $x_n^2-(a-b)x_n+1\neq 0$ (otherwise, the sequence will have period 1).

   Therefore, for such $n=k$, we must have $a+b=0$ for the first condition to be true, and hence this is a necessary condition.

2. Using the formula between x_{n+4} and x_n, we have

$$\begin{aligned} x_{n+4} &= \frac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)} \\ &= \frac{(a^2-1)\cdot\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}-(a+b)}{(a+b)\cdot\frac{(a^2-1)x_n-(a+b)}{(a+b)x_n+(b^2-1)}+(b^2-1)} \\ &= \frac{(a^2-1)\cdot\left[(a^2-1)x_n-(a+b)\right]-(a+b)\cdot\left[(a+b)x_n+(b^2-1)\right]}{(a+b)\cdot[(a^2-1)x_n-(a+b)]+(b^2-1)\cdot[(a+b)x_n+(b^2-1)]} \\ &= \frac{\left[(a^2-1)^2-(a+b)^2\right]x_n-\left[(a^2-1)(a+b)+(a+b)(b^2-1)\right]}{(a+b)\left[(a^2-1)+(b^2-1)\right]x_n+\left[(b^2-1)^2-(a+b)^2\right]}. \end{aligned}$$

   If sequence has period 4, we have $x_{n+4}=x_n$ for all integers $n\geq 0$, and the sequence does not have period 1, 2 or 3.

   We notice

$$\begin{aligned} x_{n+4}=x_n &\iff x_n\cdot\left[(a+b)\left[(a^2-1)+(b^2-1)\right]x_n+\left[(b^2-1)^2-(a+b)^2\right]\right] \\ &\qquad =\left[(a^2-1)^2-(a+b)^2\right]x_n-\left[(a^2-1)(a+b)+(a+b)(b^2-1)\right] \\ &\iff (a+b)(a^2+b^2-2)\left(x_n^2-(a-b)x_n+1\right)=0. \end{aligned}$$

   From the previous part, we know that for some $n=k\geq 0$, we must have

$$(a+b)\left(x_k^2-(a-b)x_k+1\right)\neq 0,$$

   which means $a+b\neq 0$ and $x_k^2-(a-b)x_k+1\neq 0$. Hence, we must have $a^2+b^2-2=0$.

   On the other hand, if $a^2+b^2-2=0$, $a+b\neq 0$ and $x_k^2-(a-b)x_k+1\neq 0$ for some $n=k\geq 0$, we know that the sequence does not satisfy $x_{n+1}=x_n$, does not satisfy $x_{n+2}=x_n$, and it satisfies $x_{n+4}=x_n$.

If $x_{n+3} = x_n$, then we have $x_{n+3} = x_{n+4}$ which contradicts with not satisfying $x_{n+1} = x_n$. Hence, the sequence does not satisfy $x_{n+3} = x_n$, and it must have period 4.

Therefore, the sequence has period 4, if and only if

$$\begin{cases} a + b \neq 0, \\ a^2 + b^2 - 2 = 0, \\ x_k^2 - (a - b)x_k + 1 \neq 0 \text{ for some } n = k \geq 0. \end{cases}$$