2017.2 Question 13

1. For each try, there is a probability of $\frac{1}{n}$ of getting the correct key, and $1-\frac{1}{n}$ otherwise. Let X_1 denote the number of attempts to open the door, we can see that $X_1 \sim \text{Geo}\left(\frac{1}{n}\right)$, and hence using the formula for a geometric distribution,

$$\mathrm{E}(X_1) = n.$$

The way to consider the binomial expansion is as follows. First, note the probability mass function of X_1 is

$$\mathrm{P}(X_1 = x) = \left(1 - \frac{1}{n}\right)^{x-1} \cdot \frac{1}{n},$$

and hence the expectation is given by

$$\begin{aligned}
\mathrm{E}(X_1) &= \sum_{x=1}^{\infty} x\,\mathrm{P}(X_1 = x) \\
&= \sum_{x=1}^{\infty} x \cdot \left(1 - \frac{1}{n}\right)^{x-1} \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \sum_{x=1}^{\infty} x \cdot \left(1 - \frac{1}{n}\right)^{x-1}.
\end{aligned}$$

Consider the binomial expansion of $(1-q)^{-2}$. We have

$$\begin{aligned}
(1-q)^{-2} &= \sum_{t=0}^{\infty} \frac{(-q)^t \cdot \prod_{r=1}^{t}(-2+1-t)}{t!} \\
&= \sum_{t=0}^{\infty} \frac{(-1)^t q^t (-1)^t \prod_{r=1}^{t}(1+t)}{t!} \\
&= \sum_{t=0}^{\infty} \frac{q^t (t+1)!}{t!} \\
&= \sum_{t=0}^{\infty} (t+1) q^t.
\end{aligned}$$

Let $q = 1 - \frac{1}{n}$. We can see

$$\begin{aligned}
\mathrm{E}(X_1) &= \frac{1}{n} \cdot \sum_{x=1}^{\infty} x \cdot \left(1 - \frac{1}{n}\right)^{x-1} \\
&= \frac{1}{n} \cdot \sum_{x=0}^{\infty} (x+1) \cdot q^x \\
&= \frac{1}{n} \cdot (1-q)^{-2} \\
&= \frac{1}{n} \cdot \left(\frac{1}{n}\right)^{-2} \\
&= n,
\end{aligned}$$

precisely what we had before.

2. Let X_2 be the number of attempts to open the door in this case. Considering the probability mass

function of X_2, we have for $x = 1, 2, \ldots, n$, that

$$\begin{aligned}
\mathrm{P}(X_2 = x) &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{n-(x-2)-1}{n-(x-2)} \cdot \frac{1}{n-(x-1)} \\
&= \frac{(n-1)!/(n-x)!}{n!/(n-x)!} \\
&= \frac{(n-1)!}{n!} \\
&= \frac{1}{n}.
\end{aligned}$$

This shows that X_2 follows a discrete uniform distribution on $\{1, 2, \ldots, n\}$, i.e., $X_2 \sim \mathrm{U}(n)$. Hence, $\mathrm{E}(X_2) = \frac{n+1}{2}$.

3. Let X_3 be the number of attempts to open the door in this case. Considering the probability mass function of X_2, we have for $x = 1, 2, \ldots$, that

$$\begin{aligned}
\mathrm{P}(X_3 = x) &= \frac{n-1}{n} \cdot \frac{n}{n+1} \cdots \frac{n+x-3}{n+x-2} \cdot \frac{1}{n+x-1} \\
&= \frac{(n+x-3)!/(n-2)!}{(n+x-1)!/(n-1)!} \\
&= \frac{(n+x-3)!(n-1)!}{(n+x-1)!(n-2)!} \\
&= \frac{n-1}{(n+x-1)(n+x-2)},
\end{aligned}$$

which is precisely what is desired.

By partial fractions, we have

$$\mathrm{P}(X_3 = x) = (n-1) \cdot \left(\frac{2}{n+x-2} - \frac{1}{n+x-1} \right),$$

and hence the expected number of attempts is

$$\begin{aligned}
\mathrm{E}(X_3) &= \sum_{x=1}^{\infty} (n-1) \cdot x \cdot \left(\frac{1}{n+x-2} - \frac{1}{n+x-1} \right) \\
&= (n-1) \sum_{x=1}^{\infty} x \left(\frac{1}{n+x-2} - \frac{1}{n+x-1} \right).
\end{aligned}$$

We consider the partial sum of this infinite sum op to $x = t$, and

$$\begin{aligned}
\sum_{x=1}^{t} x \left(\frac{1}{n+x-2} - \frac{1}{n+x-1} \right) &= \sum_{x=1}^{t} \frac{x}{n+x-2} - \sum_{x=1}^{t} \frac{x}{n+x-1} \\
&= \sum_{x=0}^{t-1} \frac{x+1}{n+x-1} - \sum_{x=1}^{t} \frac{x}{n+x-1} \\
&= \frac{1}{n-1} + \sum_{x=1}^{t-1} \frac{1}{n+x-1} - \frac{t}{n+t-1} \\
&= \sum_{x=0}^{t-1} \frac{1}{n+x-1} - \frac{t}{n+t-1} \\
&= \sum_{x=n-1}^{n+t-2} \frac{1}{x} - \frac{t}{n+t-1}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathrm{E}(X_3) &= (n-1)\sum_{x=1}^{\infty} x\left(\frac{1}{n+x-2} - \frac{1}{n+x-1}\right) \\
&= (n-1)\lim_{t\to\infty}\left(\sum_{x=n-1}^{n+t-2}\frac{1}{x} - \frac{t}{n+t-1}\right) \\
&= (n-1)\lim_{t\to\infty}\left(\sum_{x=1}^{n+t-2}\frac{1}{x} - \sum_{x=1}^{n-2}\frac{1}{x} - \frac{t}{n+t-1}\right) \\
&= (n-1)\left(\sum_{x=1}^{\infty}\frac{1}{x} - \sum_{x=1}^{n-2}\frac{1}{x} - 1\right)
\end{aligned}$$

does not converge since the first term (harmonic sum) diverges, and the rest of the terms are finite.