2017.2 Question 12

1. Let $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$. X and Y take values of non-negative integers. Hence, for any non-negative integer r, we have

$$\begin{aligned}
\mathrm{P}(X+Y=r) &= \sum_{t=0}^{r} \mathrm{P}(X=t, Y=r-t) \\
&= \sum_{t=0}^{r} \mathrm{P}(X=t)\,\mathrm{P}(Y=r-t) \\
&= \sum_{t=0}^{r} \frac{\lambda^t}{e^\lambda \cdot t!} \cdot \frac{\mu^{r-t}}{e^\mu \cdot (r-t)!} \\
&= \frac{1}{e^{\lambda+\mu}} \cdot \sum_{t=0}^{r} \frac{\lambda^t \mu^{r-t}}{t!(r-t)!} \\
&= \frac{1}{e^{\lambda+\mu} r!} \cdot \sum_{t=0}^{r} \frac{r!\lambda^t \mu^{r-t}}{t!(r-t)!} \\
&= \frac{1}{e^{\lambda+\mu} r!} \cdot \sum_{t=0}^{r} \binom{r}{t} \lambda^t \mu^{r-t} \\
&= \frac{1}{e^{\lambda+\mu} r!} (\lambda+\mu)^r \\
&= \frac{(\lambda+\mu)^r}{e^{\lambda+\mu} r!},
\end{aligned}$$

which is precisely the probability mass function for $\mathrm{Po}(\lambda+\mu)$, and hence $X+Y \sim \mathrm{Po}(\lambda+\mu)$.

2. We consider the probability mass function for the number of fishes Adam has caught in this situation. Given $X+Y=k$, the only values that X can take are $0, 1, \cdots, k$, and hence consider $x = 0, 1, \cdots, k$, we have

$$\begin{aligned}
\mathrm{P}(X=x \mid X+Y=k) &= \frac{\mathrm{P}(X=x, X+Y=k)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\mathrm{P}(X=x, Y=k-x)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\mathrm{P}(X=x) \cdot \mathrm{P}(Y=k-x)}{\mathrm{P}(X+Y=k)} \\
&= \frac{\frac{\lambda^x}{e^\lambda x!} \cdot \frac{\mu^{k-x}}{e^\mu (k-x)!}}{\frac{(\lambda+\mu)^k}{e^{\lambda+\mu} k!}} \\
&= \frac{\lambda^x \mu^{k-x}}{(\lambda+\mu)^k} \cdot \frac{k!}{x!(k-x)!} \\
&= \binom{k}{x} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^x \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{k-x}.
\end{aligned}$$

This is precisely the probability mass function for the binomial distribution $\mathrm{B}\left(k, \frac{\lambda}{\lambda+\mu}\right)$, and we can say that

$$(X \mid X+Y=k) \sim \mathrm{B}\left(k, \frac{\lambda}{\lambda+\mu}\right).$$

3. When the first fish is caught, this is $X+Y=1$, and $X=1$. Hence, the probability is

$$\mathrm{P}(X=1 \mid X+Y=1) = \binom{1}{1} \cdot \left(\frac{\lambda}{\lambda+\mu}\right)^1 \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{1-1} = \frac{\lambda}{\lambda+\mu}.$$

4. There is a probability of $\frac{\lambda}{\lambda+\mu}$ of Adam catching the first fish, and in this case the waiting time is first for the fish to come up (which is $\frac{1}{\lambda+\mu}$), plus the waiting time of Eve's fish to come up (which is $\frac{1}{\mu}$), summed together. This applies the other way around as well if Eve catches the first fish.

Hence, the expected time is

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\cdot\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)+\frac{\mu}{\lambda+\mu}\cdot\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\\
&=\frac{1}{\lambda+\mu}\cdot\left(\frac{\lambda}{\lambda+\mu}+\frac{\lambda}{\mu}+\frac{\mu}{\lambda+\mu}+\frac{\mu}{\lambda}\right)\\
&=\frac{1}{\lambda+\mu}\cdot\left(1+\frac{\lambda^2+\mu^2}{\lambda\mu}\right)\\
&=\frac{\lambda^2+\lambda\mu+\mu^2}{\lambda\mu(\lambda+\mu)}.
\end{aligned}$$