2017.2 Question 1

1. Using integration by parts, we notice that

$$\begin{aligned}(n+1)I_n &= (n+1)\int_0^1 x^n \arctan x \,\mathrm{d}x \\ &= \int_0^1 \arctan x \,\mathrm{d}x^{n+1} \\ &= \left[\arctan x \cdot x^{n+1}\right]_0^1 - \int_0^1 x^{n+1} \,\mathrm{d}\arctan x \\ &= \arctan 1 \cdot 1^{n+1} - \arctan 0 \cdot 0^{n+1} - \int_0^1 \frac{x^{n+1}}{1+x^2} \,\mathrm{d}x \\ &= \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} \,\mathrm{d}x.\end{aligned}$$

Set $n = 0$, and we have

$$\begin{aligned}I_0 &= (0+1)I_0 \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \,\mathrm{d}x \\ &= \frac{\pi}{4} - \frac{1}{2} \cdot \left[\ln(1+x^2)\right]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2} \cdot [\ln 2 - \ln 1] \\ &= \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

2. Using the result in the previous part,

$$\begin{aligned}(n+3)I_{n+2} + (n+1)I_n &= \left(\frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2} \,\mathrm{d}x\right) + \left(\frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2} \,\mathrm{d}x\right) \\ &= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1} + x^{n+3}}{1+x^2} \,\mathrm{d}x \\ &= \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2} \,\mathrm{d}x \\ &= \frac{\pi}{2} - \int_0^1 x^{n+1} \,\mathrm{d}x \\ &= \frac{\pi}{2} - \frac{1}{n+2}\left[x^{n+2}\right]_0^1 \\ &= \frac{\pi}{2} - \frac{1}{n+2}.\end{aligned}$$

Letting $n = 0$, and we have

$$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}.$$

Letting $n = 2$, and we have

$$5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}.$$

Subtracting the first one from the second one, and hence

$$5I_4 - I_0 = \frac{1}{4}.$$

Hence,

$$I_4 = \frac{1}{5} \cdot \left[\frac{1}{4} + \left(\frac{\pi}{4} - \frac{\ln 2}{2}\right)\right] = \frac{1}{20} + \frac{\pi}{20} - \frac{\ln 2}{10}.$$

3. Let $n = 1$, and the statement says

$$\begin{aligned}(4n+1)I_{4n} &= 5I_4 \\ &= A - \frac{1}{2}\sum_{r=1}^{2\cdot 1}(-1)^r \frac{1}{r} \\ &= A - \frac{1}{2}\left(-\frac{1}{1} + \frac{1}{2}\right) \\ &= A + \frac{1}{4}.\end{aligned}$$

Comparing to the previous expression, we claim that

$$A = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

This shows the base case for $n = 1$. For the induction step, we first introduce a lemma. Since

$$(n+5)I_{n+4} + (n+3)I_{n+2} = \frac{\pi}{2} - \frac{1}{n+4}, (n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \frac{1}{n+2},$$

subtracting the second one from the first one will give us

$$(n+5)I_{n+4} - (n+1)I_n = \frac{1}{n+2} - \frac{1}{n+4}.$$

Setting $n = 4m$, we have

$$\begin{aligned}(4(m+1)+1)I_{4(m+1)} &= (4m+1)I_{4m} + \frac{1}{4m+2} - \frac{1}{4m+4} \\ &= (4m+1)I_{4m} - \frac{1}{2}\cdot\left(-\frac{1}{2m+1} + \frac{1}{2m+2}\right) \\ &= (4m+1)I_{4m} - \frac{1}{2}\cdot\left[(-1)^{2m+1}\frac{1}{2m+1} + (-1)^{2m+2}\frac{1}{2m+2}\right].\end{aligned}$$

Now we show the inductive step. Assume the statement is true for some $n = k \geq 1$, i.e.

$$(4k+1)I_{4k} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r \frac{1}{r}.$$

Using the identity above, we have

$$\begin{aligned}(4(k+1)+1)I_{4(k+1)} &= (4k+1)I_{4k} - \frac{1}{2}\cdot\left[(-1)^{2k+1}\frac{1}{2k+1} + (-1)^{2k+2}\frac{1}{2k+2}\right] \\ &= A - \frac{1}{2}\sum_{r=1}^{2k}(-1)^r\frac{1}{r} - \frac{1}{2}\cdot\left[(-1)^{2k+1}\frac{1}{2k+1} + (-1)^{2k+2}\frac{1}{2k+2}\right] \\ &= A - \frac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r\frac{1}{r}.\end{aligned}$$

Hence, the original statement is true for $n = 1$ (as shown when determining the value of A), and given the original statement holds for some $n = k \geq 1$, it holds for $n = k + 1$. By the principle of mathematical induction, this statement holds for all $n \geq 1$, where

$$A = \frac{\pi}{4} - \frac{\ln 2}{2}.$$