2016.3 Question 8

1. If we replace x with $-x$ in the original equation, we get

$$f(-x) + (1 - (-x))f(-(-x)) = (-x)^2,$$

which simplifies to

$$f(-x) + (1 + x)f(x) = x^2$$

as desired.

Therefore, we have a pair of equations in terms of $f(x)$ and $f(-x)$:

$$\begin{cases} f(x) + (1 - x)f(-x) &= x^2 \\ (1 + x)f(x) + f(-x) &= x^2. \end{cases}$$

Multiplying the second equation by $(1 - x)$ gives us

$$(1 - x^2)f(x) + (1 - x)f(-x) = x^2(1 - x),$$

and subtracting the first equation from this

$$-x^2 f(x) = -x^3,$$

which gives $f(x) = x$.

Plugging this back, we have

$$\begin{aligned} \text{LHS} &= f(x) + (1 - x)f(-x) \\ &= x + (1 - x)(-x) \\ &= x - x + x^2 \\ &= x^2 \\ &= \text{RHS} \end{aligned}$$

which holds. Therefore, $f(x) = x$ is the solution to the functional equation.

2. For $x \neq 1$, we have

$$\begin{aligned} K(K(x)) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &= \frac{(x + 1) + (x - 1)}{(x + 1) - (x - 1)} \\ &= \frac{2x}{2} \\ &= x, \end{aligned}$$

for $x \neq 1$, as desired.

The equation on g is

$$g(x) + xg(K(x)) = x,$$

and if we substitute x as $K(x)$, we have

$$g(K(x)) + K(x)g(K(K(x))) = K(x),$$

which simplifies to

$$g(K(x)) + K(x)g(x) = K(x).$$

Multiplying the second equation by x, we have

$$xK(x)g(X) + xg(K(x)) = xK(x),$$

and subtracting the first equation from this gives

$$(xK(x)-1)g(x) = x(K(x)-1),$$

which gives

$$\begin{aligned} g(x) &= \frac{x\left(K(x)-1\right)}{xK(x)-1} \\ &= \frac{x\left(\frac{x+1}{x-1}-1\right)}{x\cdot\frac{x+1}{x-1}-1} \\ &= \frac{x\left[(x+1)-(x-1)\right]}{x(x+1)-(x-1)} \\ &= \frac{2x}{x^2+1}, \end{aligned}$$

for $x \neq 1$.

If we plug this back to the original equation, we have

$$\begin{aligned} \text{LHS} &= \frac{2x}{x^2+1} + x\frac{2\cdot\frac{x+1}{x-1}}{\left(\frac{x+1}{x-1}\right)^2+1} \\ &= \frac{2x}{x^2+1} + \frac{2x\cdot(x+1)\cdot(x-1)}{(x+1)^2+(x-1)^2} \\ &= \frac{2x}{x^2+1} + \frac{2x(x^2-1)}{2x^2+2} \\ &= \frac{2x}{x^2+1} + \frac{x(x^2-1)}{x^2+1} \\ &= \frac{x^3-x+2x}{x^2+1} \\ &= \frac{x(x^2+1)}{x^2+1} \\ &= x \\ &= \text{RHS}, \end{aligned}$$

so

$$g(x) = \frac{2x}{x^2+1}$$

is the solution to the original functional equation.

3. Let $H(x) = \frac{1}{1-x}$. Notice that

$$\begin{aligned} H(H(x)) &= \frac{1}{1-\frac{1}{1-x}} \\ &= \frac{1-x}{1-x-1} \\ &= \frac{x-1}{x} \\ &= 1-\frac{1}{x} \end{aligned}$$

and

$$\begin{aligned} H(H(H(x))) &= \frac{1}{1-\left(1-\frac{1}{x}\right)} \\ &= \frac{x}{1} \\ &= x. \end{aligned}$$

Now, if we replace all the x with $\frac{1}{1-x}$, we will get

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)=1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right),$$

and doing the same replacement again gives us

$$h\left(1-\frac{1}{x}\right)+h(x)=1-\left(1-\frac{1}{x}\right)-x.$$

Summing these two equations, together with the original equation, gives us that

$$2\cdot\left[h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)\right]=3-2\cdot\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right],$$

and therefore

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)=\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right].$$

Subtracting the second equation from this, gives that

$$\begin{aligned}h(x)&=\left(\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right]\right)-\left[1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right)\right]\\&=\frac{1}{2}-x.\end{aligned}$$

Plugging this back to the original equation, we have

$$\begin{aligned}\text{LHS}&=\frac{1}{2}-x+\frac{1}{2}-\frac{1}{1-x}\\&=1-x-\frac{1}{1-x}\\&=\text{RHS},\end{aligned}$$

which satisfies the original functional equation. Therefore, the original equation solves to

$$h(x)=\frac{1}{2}-x.$$