2016.3 Question 7

For $\omega = \exp \frac{2\pi i}{n}$, we have for $k = 0, 1, 2, \ldots, n-1$, that $\omega^k = \exp \frac{2\pi i k}{n}$. Therefore,

$$(\omega^k)^n = \exp \frac{2\pi i k n}{n} = \exp(2\pi i k) = 1.$$

Also, notice that $\arg \omega^k = \frac{2k\pi}{n}$, which means that all ω^ks are different.

This means that $\omega^0 = 1, \omega^1 = 1, \omega^2, \ldots, \omega^{n-1}$ are exactly the n roots to the polynomial $z^n - 1$, which has leading coefficient 1.

Therefore, we must have

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1,$$

as desired.

For the following parts, W.L.O.G. let the orientation of the polygon be such that $X_k = \omega^k$.

1. Let z represent the complex number for P, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= \prod_{k=0}^{n-1} \left|z - \omega^k\right| \\ &= \left|\prod_{k=0}^{n-1} (z - \omega^k)\right| \\ &= |z^n - 1|\,.\end{aligned}$$

Since P is equidistant from X_0 and X_1, we must have that $P = r \exp\left(\frac{\pi i}{n}\right)$ for some $r \in \mathbb{R}$, where $|r| = |OP|$. Therefore, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= |z^n - 1| \\ &= \left|r^n \exp\left(\frac{\pi i}{2}\right) - 1\right| \\ &= |-r^n - 1| \\ &= |r^n + 1|\,.\end{aligned}$$

If n is even, then $r^n = |r|^n > 0$, and therefore $|r^n + 1| = r^n + 1 = |r|^n + 1 = |OP|^n + 1$ as desired.

If n is odd, and $r > 0$, then $r^n = |r|^n > 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= |r|^n + 1 \\ &= |OP|^n + 1.\end{aligned}$$

When $-1 \leq r < 0$, we have $-1 \leq r^n = -|r|^n < 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= -|r|^n + 1 \\ &= -|OP|^n + 1.\end{aligned}$$

When $r < -1$, we have $r^n = -|r|^n < -1$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= -r^n - 1 \\ &= |r|^n - 1 \\ &= |OP|^n - 1.\end{aligned}$$

In summary, when n is odd, we have

$$\prod_{k=0}^{n-1} |PX_k| = \begin{cases} |OP|^n + 1, & P \text{ is in the first quadrant,} \\ -|OP|^n + 1, & P \text{ is in the third quadrant and } |OP| \leq 1, \\ |OP|^n - 1, & P \text{ is in the third quadrant and } |OP| > 1. \end{cases}$$

2. Notice that for a general point P whose complex number is z, we have

$$\begin{aligned} \prod_{k=1}^{n-1} |PX_k| &= (z-\omega)(z-\omega^2)\cdots\left(z-\omega^{n-1}\right) \\ &= \frac{z^n - 1}{z - 1} \\ &= 1 + z + z^2 + \cdots + z^{n-1}. \end{aligned}$$

If we let $P = X_0$, $z = 1$, and RHS $= n$, just as we desired.