2016.3 Question 6

- In the case where $B > A > 0$ or $-B < -A < 0$, notice that

$$R\cosh(x+\gamma) = R\cosh x \cosh\gamma + R\sinh x \sinh\gamma.$$

Therefore, we would like $R\sinh\gamma = A$ and $R\cosh\gamma = B$.
Since $\cosh\gamma^2 - \sinh\gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
We also have $\tanh\gamma = \frac{A}{B}$, and therefore $\gamma = \operatorname{artanh}\frac{A}{B}$.
Notice that $\cosh\gamma > 0$, so R must have the same sign as B.

  - If $B > A > 0$, $R = \sqrt{B^2 - A^2}$.
  - If $B < -A < 0$, $R = -\sqrt{B^2 - A^2}$.

- In the case where $-A < B < A$, notice that

$$R\sinh(x+\gamma) = R\sinh\gamma\cosh x + R\cosh\gamma\sinh x.$$

Therefore, we would like $R\cosh\gamma = A$ and $R\sinh\gamma = B$.
Since $\cosh\gamma^2 - \sinh\gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
We also have $\tanh\gamma = \frac{B}{A}$, and therefore $\gamma = \operatorname{artanh}\frac{B}{A}$.
Notice that $\cosh\gamma > 0$, so R will have the same sign as A, and hence $R = \sqrt{A^2 - B^2}$.

- When $B = A$, we have

$$\begin{aligned} A\sinh x + B\cosh x &= A\frac{e^x - e^{-x}}{2} + A\frac{e^x + e^{-x}}{2} \\ &= Ae^x. \end{aligned}$$

- When $B = -A$, we have

$$\begin{aligned} A\sinh x + B\cosh x &= A\frac{e^x - e^{-x}}{2} - A\frac{e^x + e^{-x}}{2} \\ &= Ae^{-x}. \end{aligned}$$

Therefore, in conclusion,

$$A\sinh x + B\cosh x = \begin{cases} \sqrt{B^2 - A^2}\cosh\left(x + \operatorname{artanh}\frac{A}{B}\right), & 0 < A < B, \\ Ae^x, & 0 < B = A, \\ \sqrt{A^2 - B^2}\sinh\left(x + \operatorname{artanh}\frac{B}{A}\right), & -A < B < A, \\ -Ae^{-x}, & B = -A < 0, \\ -\sqrt{B^2 - A^2}\cosh\left(x + \operatorname{artanh}\frac{A}{B}\right), & -B < -A < 0. \end{cases}$$

1. We have $\operatorname{sech} x = a\tanh x + b$, and hence $1 = a\sinh x + b\cosh x$. If $b > a > 0$, we have

$$\sqrt{b^2 - a^2}\cosh\left(x + \operatorname{artanh}\frac{a}{b}\right) = 1.$$

Therefore,

$$\begin{aligned} \cosh\left(x + \operatorname{artanh}\frac{a}{b}\right) &= \frac{1}{\sqrt{b^2 - a^2}} \\ x + \operatorname{artanh}\frac{a}{b} &= \pm\operatorname{arcosh}\frac{1}{\sqrt{b^2 - a^2}} \\ x &= \pm\operatorname{arcosh}\frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh}\frac{a}{b}, \end{aligned}$$

as desired.

2. When $a > b > 0$,
$$\sqrt{a^2 - b^2} \sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) = 1.$$
Therefore,
$$\begin{aligned}
\sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) &= \frac{1}{\sqrt{a^2 - b^2}} \\
x + \operatorname{artanh} \frac{b}{a} &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} \\
x &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} - \operatorname{artanh} \frac{b}{a}.
\end{aligned}$$

3. We would like to have two solutions to the equation $1 = a \sinh x + b \cosh x$.
   - $0 < a < b$, this gives
   $$x = \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b},$$
   For this to make sense, we must have $\frac{1}{\sqrt{b^2 - a^2}} \geq 1$, and therefore $0 < \sqrt{b^2 - a^2} \leq 1$, which is $0 < b^2 - a^2 \leq 1$.
   For this to have two distinct points, we would like to have $\operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \neq 0$ as well. This means $b^2 - a^2 \neq 1$.
   Therefore, in this case, this means that $a < b < \sqrt{a^2 + 1}$.
   - $b = a$, this gives $ae^x = 1$, which gives a unique solution $x = -\ln a$.
   - $-a < b < a$, this gives
   $$\sqrt{A^2 - B^2} \sinh \left(x + \operatorname{artanh} \frac{B}{A} \right) = 1,$$
   which can only give the solution $x = \operatorname{arsinh} \frac{1}{\sqrt{A^2 - B^2}} - \operatorname{artanh} \frac{B}{A}$.
   - $b = -a$, this gives $-ae^{-x} = 1$, which does not have a solution.
   - $-b < -a < 0$, this gives
   $$-\sqrt{b^2 - a^2} \cosh \left(x + \operatorname{artanh} \frac{a}{b} \right) = 1,$$
   but this is impossible, since both square root and cosh are always positive.

   Therefore, the only possibility is when $a < b < \sqrt{a^2 + 1}$.

4. When they touch at a point, this will mean at this value, the number of solutions will change on both sides. This is only possible when $b = \sqrt{a^2 + 1}$.
   Therefore,
   $$x = -\operatorname{artanh} \frac{a}{\sqrt{a^2 + 1}}.$$
   Hence,
   $$\begin{aligned}
   y &= a \tanh x + b \\
   &= -a \cdot \frac{a}{\sqrt{a^2 + 1}} + \sqrt{a^2 + 1} \\
   &= \frac{-a^2 + a^2 + 1}{\sqrt{a^2 + 1}} \\
   &= \frac{1}{\sqrt{a^2 + 1}}.
   \end{aligned}$$