2016.3 Question 5

1. By the binomial theorem, we have

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k.$$

If we let $x = 1$, we have

$$2^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k}.$$

Since $\binom{2m+1}{m}$ is a part of the sum, and all the other terms are positive, and there are other terms which are not $\binom{2m+1}{m}$ (e.g. $\binom{2m+1}{0} = 1$), we therefore must have

$$\binom{2m+1}{m} < 2^{2m+1}.$$

2. Notice that

$$\begin{aligned}\binom{2m+1}{m} &= \frac{(2m+1)!}{m!(m+1)!} \\ &= \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}\end{aligned}$$

A number theory argument follows. First, notice that all terms in the product $P_{m+1,2m+1}$ are within the numerator. Therefore, we must have

$$P_{m+1,2m+1} \mid (2m+1)(2m)(2m-1)\cdots(m+2).$$

Next, since all the terms in the product are primes, none of the terms will therefore have factors between 1 and m. This means that

$$\gcd\left(P_{m+1,2m+1}, m!\right) = 1,$$

i.e. $P_{m+1,2m+1}$ are coprime.

Therefore, given that $\binom{2m+1}{m} = \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}$ is an integer, we must therefore have

$$P_{m+1,2m+1} \mid \binom{2m+1}{m},$$

and hence

$$P_{m+1,2m+1} \leq \binom{2m+1}{m} < 2^{2m},$$

as desired.

3. Notice that

$$\begin{aligned}P_{1,2m+1} &= P_{1,m+1} \cdot P_{m+1,2m+1} \\ &< 4^{m+1} \cdot 2^{2m} \\ &= 4^{m+1} \cdot 4^m \\ &= 4^{2m+1},\end{aligned}$$

as desired.

4. First we look at the base case when $n = 2$.

   $P_{1,2} = 2$, $4^2 = 16$, the original statement holds when $n = 2$.

   Now, we use strong induction. Suppose the statement holds up to some $n = k \geq 2$.

   If $k = 2m$ is even, the induction step for $2m \to 2m + 1$ is already shown in the previous part.

   If $k = 2m + 1$ is odd, we must have that $k + 1$ is even. The only even prime is 2, but since $k \geq 2$, $k + 1 \neq 2$, and $k + 1$ must be composite.

   Therefore, $P_{1,k+1} = P_{1,k} < 4^k < 4^{k+1}$. This completes the induction step.

   Therefore, by strong induction, the statement $P_{1,n} < 4^n$ holds for all $n \geq 2$.