2016.3 Question 4

1. Notice that
$$\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}=\frac{x^{r+1}-x^r}{(1+x^r)(1+x^{r+1})}=\frac{x^r(x-1)}{(1+x^r)(1+x^{r+1})}.$$
Therefore, we have
$$\begin{aligned}\sum_{r=1}^{N}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\sum_{r=1}^{N}\frac{1}{x-1}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\sum_{r=1}^{N}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\left[\frac{1}{1+x}-\frac{1}{1+x^{n+1}}\right].\end{aligned}$$
For $|x|<1$, as $n\to\infty$, $x^{n+1}\to 0$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\frac{1}{x-1}\left[\frac{1}{1+x}-1\right]\\&=\frac{1}{x-1}\cdot\frac{-x}{1+x}\\&=\frac{x}{1-x^2}\end{aligned}$$
as desired.

2. Notice that
$$\begin{aligned}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=\frac{2}{e^{ry}+e^{-ry}}\cdot\frac{2}{e^{(r+1)y}+e^{-(r+1)y}}\\&=\frac{4e^{-ry-(r+1)y}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}\\&=4e^{-y}\frac{e^{-2ry}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}.\end{aligned}$$
Let $x=e^{-2y}$. We have
$$\operatorname{sech}(ry)\operatorname{sech}((r+1)y)=4e^{-y}\frac{x^r}{(1+x^r)(1+x^{r+1})}.$$
When $y>0$, $x=e^{-2y}\in(0,1)$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=4e^{-y}\frac{e^{-2y}}{1-e^{-4y}}\\&=2e^{-y}\frac{2}{e^{2y}-e^{-2y}}\\&=2e^{-y}\operatorname{cosech}(2y)\end{aligned}$$
as desired.
Notice that for all $x\in\mathbb{R}$, $\cosh x=\cosh(-x)$, therefore $\operatorname{sech} x=\operatorname{sech}(-x)$.

Therefore,

$$
\begin{aligned}
&\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=-\infty}^{0} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(-ry)\operatorname{sech}((-r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(ry)\operatorname{sech}((r-1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=2}^{+\infty} \operatorname{sech}(ry)\operatorname{sech}((r-1)y) + \operatorname{sech}(y)\operatorname{sech}(0) + \operatorname{sech}(0)\operatorname{sech}(-y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=1}^{+\infty} \operatorname{sech}((r+1)y)\operatorname{sech}(ry) + 2\operatorname{sech} y \\
&= 4e^{-y}\operatorname{cosech}(2y) + 2\operatorname{sech} y \\
&= \frac{4e^{-y}}{\sinh 2y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y}}{\sinh y \cosh y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y} + 2\sinh y}{\sinh y \cosh y} \\
&= \frac{2e^{-y} + e^{y} - e^{-y}}{\sinh y \cosh y} \\
&= \frac{e^{y} - e^{-y}}{\sinh y \cosh y} \\
&= \frac{2\cosh y}{\sinh y \cosh y} \\
&= 2\operatorname{cosech} y.
\end{aligned}
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