2016.3 Question 3

1. We have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x P(x)}{Q(x)} &= \frac{Q(x)\left[e^x P'(x)+e^x P(x)\right]-Q'(x)e^x P(x)}{Q(x)^2}\\ &= e^x\frac{\left[Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x)\right]}{Q(x)^2}\\ &= e^x\frac{x^3-2}{(x+1)^2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\left[Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x)\right]}{Q(x)^2} &= \frac{x^3-2}{(x+1)^2}\\ (x+1)^2\left[Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right).\end{aligned}$$

If we plug in $x=-1$ on both sides, we have LHS $=0$ and RHS $=Q(-1)^2\cdot(-3)$.
Therefore, $Q(-1)^2=0$, $Q(-1)=0$.
Since $Q(x)\in\mathbb{P}[x]$, we must have

$$(x+1)\mid Q(x)$$

as desired.
Therefore, $\deg Q\geq 1$, $\deg \mathrm{RHS}=3+2\deg Q$.
If $\deg P=-\infty$, $P(x)=0$,LHS $=0$ which is impossible.
If $\deg P=0$, $P(x)=C\in\mathbb{R}\setminus\{0\}$, LHS $=C(x+1)^2Q(x)$, $\deg \mathrm{LHS}=\deg q+2$, which is impossible.
Therefore, we have $\deg P'=\deg P-1$. Hence,

$$\deg Q(x)P'(x)=\deg P'(x)Q(x)=\deg P+\deg Q-1,$$

and

$$\deg Q(x)P(x)=\deg P+\deg Q.$$

Therefore,

$$\deg \mathrm{LHS}=2+\deg P+\deg Q=\deg \mathrm{RHS},$$

which gives

$$\deg P=\deg Q+1,$$

as desired.
When $Q(x)=x+1$, let $P(x)=ax^2+bx+c$ where $a\neq 0$. We have $P'(x)=2ax+b$. Therefore,

$$\begin{aligned}(x+1)^2\left[Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right)\\ Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x) &= x^3-2\\ (x+1)(2ax+b)+(x+1)(ax^2+bx+c)-(ax^2+bx+c) &= x^3-2\\ (x+1)(2ax+b)+x(ax^2+bx+c) &= x^3-2\\ ax^3+(2a+b)x^2+(2a+b+c)x+b &= x^3-2.\end{aligned}$$

This solves to $(a,b,c)=(1,-2,0)$. Therefore, $P(x)=x^2-2x$.

2. In this case, we must have that

$$(x+1)\left[Q(x)P'(x)+Q(x)P(x)-Q'(x)P(x)\right]=Q(x)^2.$$

Therefore, $Q(x)=(x+1)R(x)$ for some $R(x)\in\mathbb{P}[x]$. We may assume $P(-1)\neq 0$.
Hence, $Q'(x)=(x+1)R'(x)+R(x)$

Plugging this in gives us

$$(x+1)R(x)P'(x) + (x+1)R(x)P(x) - [(x+1)R'(x) + R(x)]\,P(x) = (x+1)R(x)^2,$$

which simplifies to

$$(x+1)\,[R(x)P'(x) + R(x)P(x) - R'(x)P(x)] - R(x)P(x) = (x+1)R(x)^2.$$

Let $x = -1$, and we can see $x+1$ divides $R(x)$, since $x+1$ can't divide $P(x)$.
Therefore, let $R(x) = (x+1)S(x)$, therefore $R'(x) = S(x) + (x+1)S'(x)$.
This gives

$$(x+1)S(x)\,[P'(x) + P(x)] - [S(x) + (x+1)S'(x)]\,P(x) - S(x)P(x) = (x+1)^2 S(x)^2,$$

which simplifies to

$$(x+1)\,[S(x)P'(x) + S(x)P(x) - S'(x)P(x)] - 2S(x)P(x) = (x+1)^2 S(x)^2.$$

Therefore, we can see that $x+1$ divides $S(x)$ by similar reasons.
Repeating this, we can conclude that there are arbitrarily many factors of $x+1$ in $Q(x)$ (proof by infinite descent), which is impossible.
Formally speaking, let $Q(x) = (x+1)^n T(x)$ where $T(-1) \neq 0$, $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} Q'(x) &= n(x+1)^{n-1}T(x) + (x+1)^n T'(x) \\ &= (x+1)^{n-1}\left[nT(x) + (x+1)T'(x)\right]. \end{aligned}$$

Therefore,

$$(x+1)\,[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] = Q(x)^2$$

simplifies to

$$(x+1)^{n+1}T(x)\,[P'(x) + P(x)] - (x+1)^n\,[nT(x) + (x+1)T'(x)]\,P(x) = (x+1)^{2n}T(x)^2,$$

which further simplifies to

$$(x+1)\,[T(x)P'(x) + T(x)P(x) - T'(x)P(x)] - nT(x)P(x) = (x+1)^n T(x)^2.$$

Now, let $x = -1$, we have that $nT(-1)P(-1) = 0$. But $n \neq 0$, $T(-1) \neq 0$, $P(-1) \neq 0$, which gives a contradiction.
Therefore, such P and Q do not exist.