2016.3 Question 2

1. For $y^2 = 4ax$, we have $x = \frac{y^2}{4a}$, and therefore
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2y}{4a}.$$
Therefore, the normal through Q, l_Q satisfies that
$$l_Q : x - aq^2 = -\frac{4a}{2 \cdot 2aq} \cdot (y - 2aq),$$
i.e.
$$l_Q : q(x - aq^2) = -(y - 2aq).$$
Since $P \in l_Q$, we must have
$$\begin{aligned} q(ap^2 - aq^2) &= -(2ap - 2aq) \\ aq(p+q)(p-q) &= -2a(p-q) \\ pq + q^2 &= -2 \\ q^2 + pq + 2 &= 0 \end{aligned}$$
as desired.

2. We also have
$$r^2 + pr + 2 = 0.$$
Since $q \neq r$, q, r are the solutions to the equation
$$x^2 + px + 2 = 0,$$
and therefore $q + r = -p, qr = 2$.
Note that the equation for QR satisfies that
$$m_{QR} = \frac{2ar - 2aq}{ar^2 - aq^2} = \frac{2}{r+q}.$$
Therefore, l_{QR} satisfies that
$$\begin{aligned} l_{QR} : y - 2aq &= \frac{2}{r+q}(x - aq^2) \\ y &= \frac{2}{r+q}\left(x - aq^2 + \frac{r+q}{2} \cdot 2aq\right) \\ y &= \frac{2}{r+q}\left(x - aq^2 + aq^2 + aqr\right) \\ y &= \frac{2}{r+q}\left(x + aqr\right) \\ y &= -\frac{2}{p}(x + 2a). \end{aligned}$$
This passes through a fixed point $(-2a, 0)$.

3. OP has equation $y = \frac{2ap}{ap^2}x$, which is $y = \frac{2x}{p}$. Therefore, since $T = OP \cap QR$, x_T must satisfy that
$$\begin{aligned} -\frac{2}{p}(x + 2a) &= \frac{2x}{p}, \\ -(x + 2a) &= x \\ x &= -a. \end{aligned}$$
Therefore, $y_T = -\frac{2a}{p}$, $T\left(-a, -\frac{2a}{p}\right)$ lies on the line $x = -a$ which is independent of p.

The distance from the x-axis to T is $\left|\frac{2a}{p}\right| = \frac{2a}{|p|}$.

Notice that since $qr = 2$, q and r must take the same parity, and therefore $|p| = |q| + |r|$. By the AM-GM inequality, we have

$$|q| + |r| \geq 2\sqrt{|q| \cdot |r|} = 2\sqrt{2},$$

with the equal sign holding if and only if $|q| = |r|$, $q = r$, which is impossible.

Therefore, $|p| > 2\sqrt{2}$ and therefore $\frac{2a}{|p|} < \sqrt{2}$ as desired.