2016.3 Question 13

For a random variable X with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, we have

$$\kappa(X) = \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3$$

We have $Y = X - a$. Therefore, $\mathrm{E}(Y) = \mu - a$ and $\mathrm{Var}(Y) = \sigma^2$.

$$\begin{aligned}
\kappa(Y) &= \frac{\mathrm{E}\left[(Y-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[((X-a)-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3 \\
&= \kappa(X),
\end{aligned}$$

as desired.

1. Let $X \sim \mathrm{N}(0, \sigma^2)$, $\mu = 0$. Notice that

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3.$$

X has p.d.f.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Therefore,

$$\mathrm{E}(X^4) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x.$$

Now, consider using integration by parts. Notice that

$$\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x,$$

and therefore, using integration by parts, we have

$$\begin{aligned}
&\int x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x \\
&= -\sigma^2 \int x^3 \,\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) \\
&= -\sigma^2 \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) - \int \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}(x^3)\right] \\
&= 3\sigma^2 \int x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x - \sigma^2 x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right).
\end{aligned}$$

Therefore, considering the definite integral, we have

$$\begin{aligned}
\mathrm{E}(X^4) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[3\int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x - \left[x^3\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{-\infty}^{+\infty}\right] \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[3\cdot\sigma\sqrt{2\pi}\cdot\sigma^2 - 0\right] \\
&= 3\sigma^4.
\end{aligned}$$

Therefore,

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0,$$

as desired.

An alternative solution exists using generating functions.

Recall that a general normal distribution $\mathrm{N}(\mu, \sigma^2)$ has MGF

$$M(t) = \exp(\mu t + \frac{\sigma^2}{2} t^2),$$

and hence

$$\begin{aligned} M_X(t) &= \exp\left(\frac{\sigma^2}{2} t^2\right) \\ &= 1 + \left(\frac{\sigma^2}{2} t^2\right) + \frac{\left(\frac{\sigma^2}{2} t^2\right)}{2!} + \dots. \end{aligned}$$

Therefore,

$$\mathrm{E}(X^4) = M_X^{(4)}(0) = \left(\frac{\sigma^2}{2}\right)^4 \cdot 4! = 3\sigma^4,$$

and the result follows.

2. Notice that

$$\begin{aligned} & T^4 \\ &= \sum_a \binom{4}{4} Y_a^4 + \sum_{a<b} \left[\binom{4}{1,3} Y_a Y_b^3 + \binom{4}{2,2} Y_a^2 Y_b^2\right] \\ &\quad + \sum_{a<b<c} \binom{4}{1,1,2} Y_a Y_b Y_c^2 + \sum_{a<b<c<d} \binom{4}{1,1,1,1} Y_a Y_b Y_c Y_d \\ &= \sum_a Y_a^4 + \sum_{a<b} (4Y_a Y_b^3 + 6Y_a^2 Y_b^2) + \sum_{a<b<c} 12 Y_a Y_b Y_c^2 + \sum_{a<b<c<d} 24 Y_a Y_b Y_c Y_d, \end{aligned}$$

where

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! a_2! \dots a_k!}, \sum_{i=1}^k a_i = n$$

stands for the multinomial coefficient.

Note that $\mathrm{E}(Y_r) = 0$ for any $r = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} \mathrm{E}(Y_a Y_b^3) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b^3) = 0, \\ \mathrm{E}(Y_a Y_b Y_c^2) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c^2) = 0, \\ \mathrm{E}(Y_a Y_b Y_c Y_d) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c)\,\mathrm{E}(Y_d) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}(T^4) &= \sum_a \mathrm{E}(Y_a^4) + \sum_{a<b} 6\,\mathrm{E}(Y_a^2 Y_b^2) \\ &= \sum_{r=1}^n \mathrm{E}(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2), \end{aligned}$$

as desired.

3. Let $Y_i = X_i - \mu$ for $i = 1, 2, \dots, n$, and $\mu = \mathrm{E}(X), \sigma^2 = \mathrm{Var}(X) = \mathrm{Var}(Y)$ with $\mathrm{E}(Y) = 0$
Therefore, let $T = \sum_i^n Y_i = \sum_i^n X_i - n\mu$, we must have $\mathrm{E}(T) = 0$ and $\mathrm{Var}(T) = n\sigma^2$.

But since the kurtosis remains constant with shifts, we must have that $\kappa(Y_i) = \kappa$, and

$$\kappa(T) = \kappa\left[\sum_i^n X_i\right].$$

Hence, we have

$$\begin{aligned}
\kappa\left[\sum_i^n X_i\right] &= \kappa(T) \\
&= \frac{\mathrm{E}(T^4)}{(n\sigma^2)^2} - 3 \\
&= \frac{\sum_{r=1}^n \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2)}{n^2\sigma^4} - 3 \\
&= \frac{1}{n^2}\sum_{r=1}^n \frac{\mathrm{E}(Y_r^4)}{\sigma^4} + \frac{6}{n^2}\sum_{r=1}^{n-1}\sum_{s=r+1}^n \frac{\sigma^4}{\sigma^4} - 3 \\
&= \frac{1}{n^2} n \cdot (\kappa + 3) + \frac{6}{n^2}\binom{n}{2} - 3 \\
&= \frac{\kappa}{n} + \frac{3n + 3n(n-1) - 3n^2}{n^2} \\
&= \frac{\kappa}{n} + 0 \\
&= \frac{\kappa}{n},
\end{aligned}$$

as desired.