2016.3 Question 12

1. Let $X \sim \mathrm{B}(100n, 0.2)$. We have $\mu = 100n \cdot 0.2 = 20n$, and $\sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n$.

   We have that

$$\begin{aligned}\alpha &= \mathrm{P}(16n \le X \le 24n)\\ &= \mathrm{P}(|(X - 20n)| \le 4n)\\ &= \mathrm{P}(|(X-\mu)| \le \sigma\sqrt{n})\\ &= 1 - \mathrm{P}(|(X-\mu)| > \sigma\sqrt{n})\\ &\ge 1 - \frac{1}{\sqrt{n}^2}\\ &= 1 - \frac{1}{n},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2. Let $X \sim \mathrm{Po}(n)$. Therefore, $\mu = \mathrm{E}(X) = n$, $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{n}$. To show the desired inequality is equivalent to showing that

$$\frac{1 + n + \frac{n^2}{2!} + \cdot + \frac{n^{2n}}{(2n!)}}{e^n} \ge 1 - \frac{1}{n}.$$

   Notice that the left-hand side is simply $\mathrm{P}(0 \le X \le 2n)$. By the Chebyshev Inequality, we have

$$\begin{aligned}\mathrm{LHS} &= \mathrm{P}(0 \le X \le 2n)\\ &= \mathrm{P}(|X-\mu| \le n)\\ &= \mathrm{P}(|X-\mu| \le \sqrt{n}\sigma)\\ &= 1 - \mathrm{P}(|X-\mu| > \sqrt{n}\sigma)\\ &\ge 1 - \frac{1}{n}\\ &= \mathrm{RHS},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.