2016.3 Question 1

Notice that

$$I_n = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+2ax+b)^n} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{((x+a)^2+(b-a^2))^n}.$$

1. Let $x + a = \sqrt{b-a^2}\tan u$. When $x \to -\infty$, $u \to -\frac{\pi}{2}$, and when $x \to +\infty$, $u \to \frac{\pi}{2}$. We have also

$$\begin{aligned} \mathrm{d}x = \mathrm{d}(x+a) &= \mathrm{d}\sqrt{b-a^2}\tan u \\ &= \sqrt{b-a^2}\,\mathrm{d}\tan u \\ &= \sqrt{b-a^2}\sec^2 u\,\mathrm{d}u. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x+a)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{\left(\sqrt{b-a^2}\tan u\right)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{(b-a^2)(\tan^2 u+1)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u\,\mathrm{d}u}{\sqrt{b-a^2}\sec^2 u} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sqrt{b-a^2}} \\ &= \frac{\pi}{\sqrt{b-a^2}}, \end{aligned}$$

as desired.

2. Using the same substitution, we have

$$\begin{aligned} I_n &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{[(x+a)^2+(b-a^2)]^n} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} \\ &= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}. \end{aligned}$$

Therefore,

$$2n(b-a^2)I_{n+1} = (2n-1)I_n,$$

is equivalent to

$$2n\sqrt{b-a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\frac{1}{\sqrt{b-a^2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

is equivalent to

$$2n(b-a^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

is equivalent to

$$2n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u}.$$

Notice that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\tan u}{\sec^{2n} u} \\
&= \lim_{\substack{a \to \frac{\pi}{2} \\ b \to -\frac{\pi}{2}}} \left[\frac{\tan u}{\sec^{2n} u}\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan u \,\mathrm{d}\sec^{-2n} u \\
&= \lim_{\substack{a \to \frac{\pi}{2} \\ b \to -\frac{\pi}{2}}} \left[\sin u \cos^{2n-1} u\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2n \sec u \tan u \sec^{-2n-1} u \tan u \,\mathrm{d}u \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sec^2 u - 1) \,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} - 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u}.
\end{aligned}$$

This means

$$(2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} = 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u},$$

which is exactly what was desired.

3. Proof by induction:

- **Base Case.** When $n = 1$,

$$\text{LHS} = I_1 = \frac{\pi}{\sqrt{b - a^2}},$$

$$\text{RHS} = \frac{\pi}{2^{2\cdot 1-2}(b-a^2)^{1-\frac{1}{2}}} \binom{2\cdot 1 - 2}{1 - 1} = \frac{\pi}{\sqrt{b-a^2}} \binom{0}{0} = \frac{\pi}{\sqrt{b-a^2}}.$$

- **Induction Hypothesis.** Assume for some $n = k \in \mathbb{N}$, we have

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

- **Induction Step.** When $n = k + 1$,

$$\begin{aligned}
I_n &= I_{k+1} \\
&= \frac{2k+1}{2(k+1)(b-a^2)} I_k \\
&= \frac{2k+1}{2(k+1)(b-a^2)} \cdot \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}} \binom{2k-2}{k-1} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{(2k-2)!}{(k-1)!(k-1)!} \frac{(2k+1)(2k+2)}{(k+1)^2} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{2k!}{k!k!} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \binom{2k}{k} \\
&= \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.
\end{aligned}$$

Therefore, by the principle of mathematical induction, for $n \in \mathbb{N}$,

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1},$$

as desired.