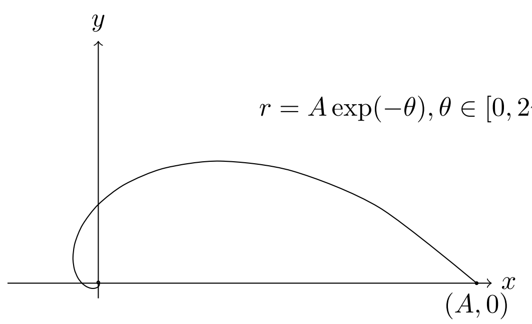
2015.3 Question 8

1. First, notice that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{\mathrm{d}r/\mathrm{d}\theta \cdot \sin\theta + r \cdot \cos\theta}{\mathrm{d}r/\mathrm{d}\theta \cdot \cos\theta - r \cdot \sin\theta}.$$
Therefore, the original differential equation reduces to
$$(r\sin\theta + r\cos\theta)\frac{\mathrm{d}r/\mathrm{d}\theta \cdot \sin\theta + r \cdot \cos\theta}{\mathrm{d}r/\mathrm{d}\theta \cdot \cos\theta - r \cdot \sin\theta} = r\sin\theta - r\cos\theta$$
which further reduces to (since $r \neq 0$)
$$(\sin\theta + \cos\theta)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \sin\theta + r\cos\theta\right] = (\sin\theta - \cos\theta)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \cos\theta - r\sin\theta\right].$$
Expanding the brackets and cancelling the equivalent terms gives us
$$r\cos^2\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta = -\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta - r\sin^2\theta,$$
which reduces to (due to the Pythagoras Theorem $\sin^2\theta + \cos^2\theta = 1$),
$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0,$$
as desired.
The rearrangement (since $r \neq 0$)
$$\frac{\mathrm{d}r}{r} = -\,\mathrm{d}\theta$$
shows that the solution to this differential equation must satisfy that (since $r > 0$)
$$\ln r = -\theta + C,$$
i.e.
$$r = A\exp(-\theta),$$
where $A > 0$.
For critical values, notice that when $\theta = 0$, $r = A$, and when $\theta = 2\pi$, $r = \frac{A}{\exp 2\pi}$, and that r is decreasing with θ. The graph will look like a spiral
A sketch is shown below, for $\theta \in [0, 2\pi)$.



2. Similar to the previous part, the equation reduces to
$$\left(\sin\theta + \cos\theta - \cos\theta \cdot r^2\right)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \sin\theta + r\cos\theta\right] = \left(\sin\theta - \cos\theta - \sin\theta \cdot r^2\right)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \cos\theta - r\sin\theta\right],$$
and hence, by expanding brackets and eliminating terms,
$$\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta + r\cos^2\theta - r^3\cos^2\theta = -r\sin^2\theta - \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta + r^3\sin^2\theta,$$

which then simplifies to
$$\frac{\mathrm{d}r}{\mathrm{d}\theta}+r-r^3=0.$$
Notice that $r=1$ is a solution to this differential equation. Therefore, rearranging terms, we have
$$\frac{\mathrm{d}r}{r^3-r}=\mathrm{d}\theta.$$
By partial fractions
$$\frac{1}{r^3-r}=-\frac{1}{r}+\frac{1}{2(r+1)}+\frac{1}{2(r-1)},$$
we therefore must have
$$\left[-\frac{1}{r}+\frac{1}{2(r+1)}+\frac{1}{2(r-1)}\right]\cdot\mathrm{d}r=\mathrm{d}\theta.$$
This therefore means that
$$\frac{1}{2}\ln|r+1|+\frac{1}{2}\ln|r-1|-\ln|r|=\theta+C,$$
for some constant $C\in\mathbb{R}$.

Combining logarithms and absolute values gives us
$$\ln\left|\frac{r^2-1}{r^2}\right|=2\theta+C,$$
and therefore,
$$\frac{r^2-1}{r^2}=\pm\exp C\cdot\exp(2\theta),$$
and this can be simplified to
$$1-\frac{1}{r^2}=\pm\exp C\cdot\exp(2\theta),$$
and therefore
$$r^2=\frac{1}{1\mp\exp C\cdot\exp(2\theta)}.$$
Let $A=\mp\exp C\neq 0$, and therefore
$$r^2=\frac{1}{1+A\exp(2\theta)}.$$
Notice when $r=1$, r satisfies that
$$r^2=1,$$
so the general solution will be
$$r^2=\frac{1}{1+A\exp(2\theta)}$$
for $A\in\mathbb{R}$ which this equation makes sense.

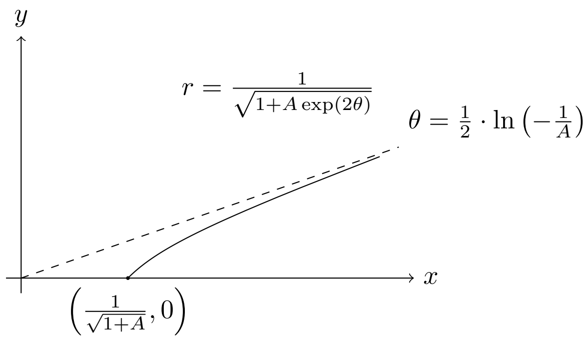
We restrict ourselves to $\theta\in[0,2\pi)$.

Notice that, this equation makes sense for all $A\geq 0$, since the denominator is obviously non-negative.

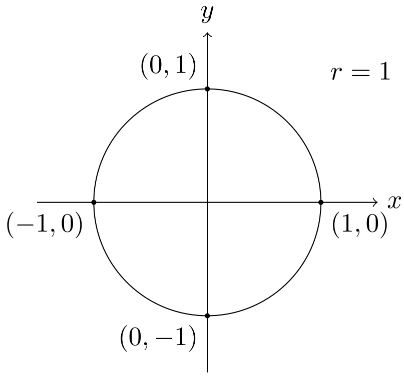
For $A<0$, the denominator is decreasing in θ, and we would like it to be greater than zero for some $\theta\in[0,2\pi)$. Therefore, we would like the maximum possible value of the denominator to be greater than, that is when $\theta=0$:
$$1+A\exp 0>0,$$
which gives $A>-1$.

We consider three cases where $r>0$, i.e.,
$$r=\frac{1}{\sqrt{1+A\exp(2\theta)}}.$$
Notice this always passes through $\left(\frac{1}{\sqrt{1+A}},0\right)$.

- When $-1 < A < 0$, the curve is not defined for
$$1 + A\exp(2\theta) \leq 0,$$
and this is precisely when
$$\exp 2\theta \geq -\frac{1}{A},$$
which is
$$\theta \geq \frac{1}{2} \cdot \ln\left(-\frac{1}{A}\right).$$
This means the curve will have an asymptote of line
$$\theta = \frac{1}{2} \cdot \ln\left(-\frac{1}{A}\right).$$
Also note that r is increasing in θ in this case, and $r \to \infty$ as $\theta \to$ the asymptote.



- When $A = 0$, notice this just gives $r = 1$, which is a circle with radius 1 centred at the origin.



- In the final case where $A > 0$, the following case arises.