2015.3 Question 7

Note that

$$\begin{aligned} D^2 x^a &= x \frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} x^a \right) \\ &= x \frac{\mathrm{d}}{\mathrm{d}x} \left(x \cdot a x^{a-1} \right) \\ &= ax \frac{\mathrm{d}}{\mathrm{d}x} x^a \\ &= ax \cdot a \cdot x^{a-1} \\ &= a^2 x^a, \end{aligned}$$

as desired.

1. Since we have that $\mathrm{d}x^a = x \cdot x^{a-1} \cdot a = ax^a$, and a is just a constant, then we must have

   $$D^n x^a = a^n x^a.$$

   If $P(x)$ is a polynomial of degree r, let

   $$P(x) = \sum_{k=0}^{r} t_k x^k.$$

   Therefore,

   $$D^n P(x) = \sum_{k=0}^{r} k^n t_k x^k.$$

   Notice that the highest degree term is $r^n t_r x^r$.

   Since $P(x)$ originally has degree $r \geq 1$, we have $r \neq 0$ and $t_r \neq 0$, and therefore this term is non-zero.

   This implies $D^n P(x)$ has degree r as well.

2. We show this by induction on n. The base case where $n = 0$ is trivially true if we define D^0 as the identity. Now, assume this is true for some $n = k < m - 1$, i.e.

   $$D^k (1-x)^m = (1-x)^{m-k} \cdot Q(x)$$

   for some polynomial Q, we aim to show this for $n = k + 1 < m$. We have

   $$\begin{aligned} D^{k+1}(1-x)^m &= D\left[(1-x)^{m-k} \cdot Q(x)\right] \\ &= x\left[-(m-k)(1-x)^{m-k-1} Q(x) + (1-x)^{m-k} Q'(x)\right] \\ &= (1-x)^{m-k-1} x \left[-(m-k)Q(x) + (1-x)Q'(x)\right], \end{aligned}$$

   which shows $D^{k+1}(1-x)^m$ is divisible by $(1-x)^{m-k-1}$ which finishes our induction step. Hence, by the principle of mathematical induction, the original statement holds for any $n < m$.

3. Notice that

   $$(1-x)^m = \sum_{r=0}^{m} \binom{m}{r} (-x)^r,$$

   and hence

   $$D^n (1-x)^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n x^r.$$

   Evaluate this at $x = 1$, we can see

   $$\left[D^n (1-x)^m\right]_{x=1} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n 1^r = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n.$$

But for $n < m$, $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$ and hence by $(1-x)$. This means that

$$[D^n(1-x)^m]_{x=1} = 0.$$

Hence,

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0,$$

as desired.