2015.3 Question 6

1. - **Only-if direction.** If w, z are real, then $u = w + z$ and $v = w^2 + z^2$ are real. Also,

$$\begin{aligned}2v - u^2 &= 2(w^2 + z^2) - (w + z)^2 \\ &= w^2 - 2wz + z^2 \\ &= (w - z)^2 \\ &\geq 0,\end{aligned}$$

   which implies $u^2 \leq 2v$ as desired.

   - **If direction.** If $u, v \in \mathbb{R}$ and $u^2 \leq 2v$, we notice that

$$wz = \frac{u^2 - v}{2} \in \mathbb{R}.$$

   Hence, w, z are solutions to the quadratic equation

$$x^2 - ux + \frac{u^2 - v}{2} = 0.$$

   Notice all coefficients in this equation is real. The discriminant satisfies

$$\begin{aligned}\Delta &= (-u)^2 - 4 \cdot 1 \cdot \frac{u^2 - v}{2} \\ &= u^2 - 2(u^2 - v) \\ &= 2v - u^2 \\ &\geq 0,\end{aligned}$$

   which implies both solutions must be real, i.e. w, z are real, as desired.

2. By simplification, we notice that letting $u = w + z$ and $v = w^2 + z^2$, we have

$$\begin{aligned}w^3 + z^3 &= (w + z)(w^2 + z^2) - wz(w + z) \\ &= (w + z)(w^2 + z^2) - \frac{1}{2}((w + z)^2 - (w^2 + z^2))(w + z) \\ &= uv - \frac{u(u^2 - v)}{2} \\ &= u\left(v - \frac{u^2 - v}{2}\right) \\ &= \frac{u}{2}\left(2v - (u^2 - v)\right) \\ &= \frac{u(3v - u^2)}{2}.\end{aligned}$$

   This means,

$$-\lambda + \lambda u = \frac{u\left[3 \cdot \left(u^2 - \frac{2}{3}\right) - u^2\right]}{2}$$

   which simplifies to

$$(u - 1)(u^2 + u - \lambda) = 0.$$

   Therefore, $u_1 = 1$. The discriminant of the remaining quadratic is

$$\Delta = 1 + 4\lambda > 1 > 0,$$

   since $\lambda > 0$.

   Therefore, u must always be real.

   The only case where there are less than 3 possible values of u, is when $u_1 = 1$ is also a solution to the quadratic.

This is precisely when $\lambda = u^2 + u = 1^2 + 1 = 2$.

Apart from this case, the two real solutions to the quadratic are distinct and must not be 1, and there are three real values of u.

Since u is always real, $u = w + z$ is always real and $v = w^2 + z^2$ is always real. However, notice that

$$2v - u^2 = 2 \cdot \left(u^2 - \frac{2}{3}\right) - u^2 = u^2 - \frac{4}{3}.$$

But when $u = 1$, $2v - u^2 < 0$, $2v < u^2$, and by part (i) at least one of w, z is not real.