2015.3 Question 4

1. Let $f(z) = z^3 + az^2 + bz + c$. If we restrict the domain to the reals, we have

$$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty.$$

By the definition of a limit, this means that $f(x) > 0$ for sufficiently big xs (say, for all $x \geq A$), and $f(x) < 0$ for sufficiently small xs (say, for all $x \leq B$).

Since f is continuous on $[B, A] \subset \mathbb{R}$, and $f(B) < 0$, $f(A) > 0$. This means that for some $\xi \in (B, A) \subset \mathbb{R}$ such that $f(\xi) = 0$, which finishes our proof.

2. By Vieta's Theorem, we have

$$\begin{aligned} z_1 + z_2 + z_3 &= -a, \\ z_1z_2 + z_1z_3 + z_2z_3 &= b, \\ z_1z_2z_3 &= -c. \end{aligned}$$

Therefore, we have $S_1 = -a$ and $a = -S_1$. Notice that

$$\begin{aligned} \frac{S_1^2 - S_2}{2} &= \frac{(z_1 + z_2 + z_3)^2 - (z_1^2 + z_2^2 + z_3^2)}{2} \\ &= \frac{2 \cdot (z_1z_2 + z_1z_3 + z_2z_3)}{2} \\ &= z_1z_2 + z_1z_3 + z_2z_3 \\ &= b. \end{aligned}$$

This means

$$\begin{aligned} a &= -S_1, \\ b &= \frac{S_1^2 - S_2}{2}. \end{aligned}$$

Also, notice that

$$\begin{aligned} -S_1^3 + 3S_1S_2 - 2S_3 &= -(z_1 + z_2 + z_3)^3 + 3(z_1 + z_2 + z_3)(z_1^2 + z_2^2 + z_3^2) - 2(z_1^3 + z_2^3 + z_3^3) \\ &= -(z_1^3 + z_2^3 + z_3^3 + 3z_1z_2^2 + 3z_1z_3^2 + 3z_2z_1^2 + 3z_2z_3^2 + 3z_3z_1^2 + 3z_3z_2^2 + 6z_1z_2z_3) \\ &\quad + 3(z_1^3 + z_2^3 + z_3^3 + z_1z_2^2 + z_1z_3^2 + z_2z_1^2 + z_2z_3^2 + z_3z_1^2 + z_3z_2^2) \\ &\quad - 2(z_1^3 + z_2^3 + z_3^3) \\ &= -6z_1z_2z_3 \\ &= 6c, \end{aligned}$$

as desired.

3. Consider the complex numbers $z_k = r_k \exp(i\theta_k)$ for $k = 1, 2, 3$.

This means that $z_k^n = r_k^n \exp(in\theta_k)$ by de Moivre's theorem, hence

$$\operatorname{Im} z_k^n = r_k^n \sin(n\theta_k).$$

This converts our condition to

$$\operatorname{Im} \sum_{k=1}^{3} z_k^n = 0$$

for $n = 1, 2, 3$.

Therefore, S_1, S_2, S_3 are real, and therefore, so are a, b, c.

Hence, by part (i), there must be some k such that z_k is real, which means θ_k is some multiple of 2π.

Since $\theta_k \in (-\pi, \pi)$, we must have $\theta_k = 0$ for such.

If $\theta_1 = 0$, $z_1 \in \mathbb{R}$. This therefore means that $z_1^n \in \mathbb{R}$, and hence

$$\operatorname{Im} \sum_{k=2}^{3} z_k^n = 0$$

for $n = 1, 2, 3$.

Consider the polynomial $(z - z_2)(z - z_3) = 0$, and let the expansion be $z^2 + pz + q = 0$.

By Vieta's Theorem, we have

$$\begin{aligned} z_2 + z_3 &= -p, \\ z_2 z_3 &= q. \end{aligned}$$

This therefore means that

$$\begin{aligned} p &= -(z_2 + z_3), \\ 2q &= (z_2 + z_3)^2 - (z_2^2 + z_3^2). \end{aligned}$$

If $z_2+z_3 \in \mathbb{R}$ and $z_2^2+z_3^2 \in \mathbb{R}$, then $p, q \in \mathbb{R}$, and z_2, z_3 are solutions to a real quadratic (polynomial).

Hence, the first case is z_2, z_3 are both real, which gives $\theta_2 = \theta_3 = 0$ since $r_k > 0$, and hence $\theta_2 = -\theta_3$.

The other case where z_2, z_3 are complex congruent to each other gives $\theta_2 = -\theta_3 + 2k\pi$ where $k \in \mathbb{Z}$ due to $r_k > 0$. But since $\theta_2, \theta_3 \in (-\pi, \pi)$, it must be the case that $\theta_2 = -\theta_3$, since the width of the interval is exactly 2π, and it is an open interval.

This finishes our proof.