2015.3 Question 3

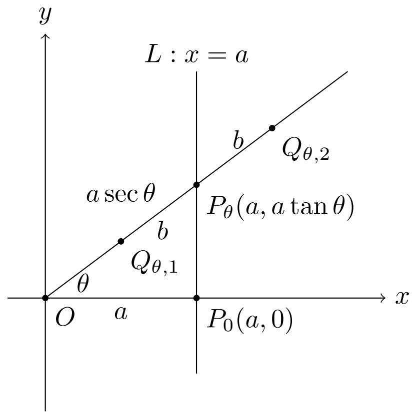
1. We prove the first part by contradiction. Assume that $\sec\theta \not> 0$, this means $\sec\theta \leq -1$.

   But in this case,
   $$r - a\sec\theta \geq r + a \geq a > b,$$
   but $|r - a\sec\theta| = b$, implies $r - a\sec\theta \leq b$, and this leads to a contradiction.

   This implies that $\sec\theta > 0$. Hence, $\cos\theta > 0$, and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   We aim to show that $|r - a\sec\theta| = b$ lies on the conchoid of Nicomedes where $L : x = a$ and $d = b$, with $A(0, 0)$.

   Let O be the origin, $P_\theta(a, a\tan\theta)$ and $P_0(a, 0)$. All points on the half-line OP_θ will have argument θ.

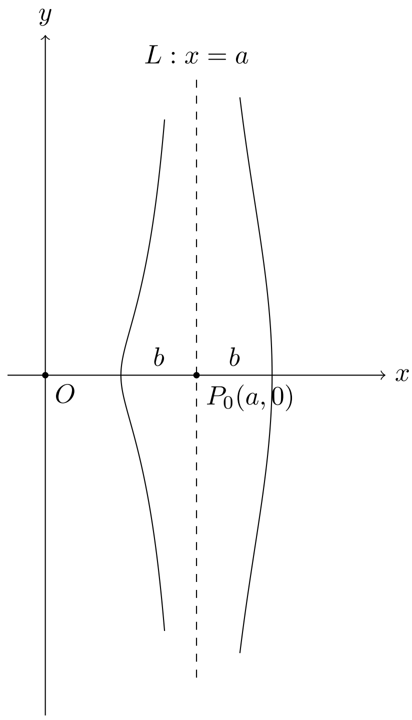
   Let Q_θ be the points on such line, satisfying the given equation $|r - a\sec\theta| = b$.

   For every $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have
   $$|OP_\theta| = |OP_0|\sec\theta = a\sec\theta.$$

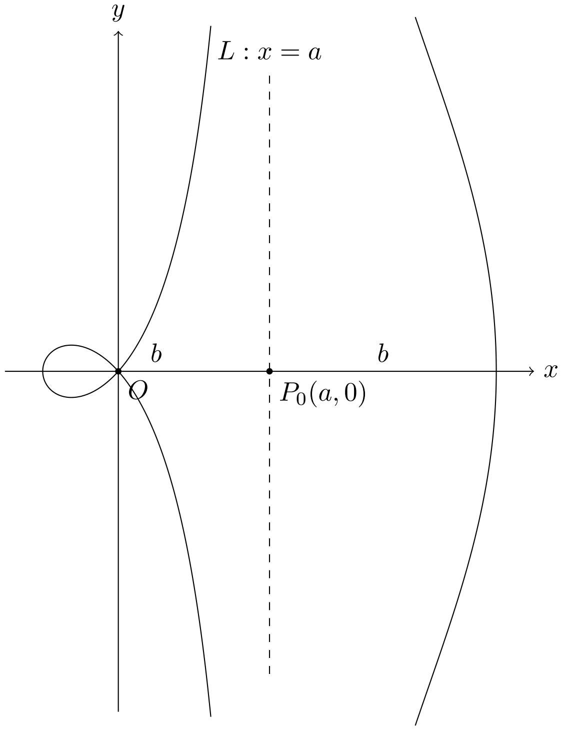
   The given equation $|r - a\sec\theta| = b$ simplifies to $r = a\sec\theta \pm b$.

   This implies that Q_θ must lie on the half-line OP_θ through O, and a fixed distance b away measured along OP_θ from line $L : x = a$ (which is measured from P_θ).

   This is precisely the definition of a conchoid of Nicomedes, and this finishes our proof.

2. The sketch is as below.



When $\sec\theta < 0$, $\sec\theta \leq -1$. We have $r = a\sec\theta \pm b$.
Since $r \geq 0$, we must have $r = a\sec\theta + b \geq 0$ (since if $r = a\sec\theta - b$, then $r < 0$), and hence

$$-1 \geq \sec\theta \geq -\frac{b}{a}, -1 \leq \cos\theta \leq -\frac{a}{b},$$

which means the area of the loop is given by the range of

$$\theta \in \left(-\pi, -\arccos\left(-\frac{a}{b}\right)\right] \cup \left[\arccos\left(-\frac{a}{b}\right), \pi\right].$$

Therefore, the area of the loop is given by

$$A = \frac{1}{2}\left[\int_{-\pi}^{-\arccos\left(-\frac{a}{b}\right)} r^2 \, \mathrm{d}\theta + \int_{\arccos\left(-\frac{a}{b}\right)}^{\pi} r^2 \, \mathrm{d}\theta\right].$$

Notice that

$$\begin{aligned}\int r^2 \, \mathrm{d}\theta &= \int \left(a^2\sec^2\theta + 2ab\sec\theta + b^2\right) \mathrm{d}\theta \\ &= a^2\tan\theta + 2ab\ln|\sec\theta + \tan\theta| + b^2\theta + C \\ &= \tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta + C,\end{aligned}$$

and

$$\alpha = \arccos\left(-\frac{a}{b}\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

Therefore,

$$\begin{aligned}
A &= \frac{1}{2}\left[\int_{-\pi}^{-\arccos\left(-\frac{a}{b}\right)} r^2 \,\mathrm{d}\theta + \int_{\arccos\left(-\frac{a}{b}\right)}^{\pi} r^2 \,\mathrm{d}\theta\right] \\
&= \frac{1}{2}\left[(\tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta)_{-\pi}^{-\frac{2\pi}{3}} + (\tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta)_{\frac{2\pi}{3}}^{\pi}\right] \\
&= \frac{1}{2}\left[\left(\sqrt{3} + 4\ln\left|-2+\sqrt{3}\right| - \frac{8\pi}{3}\right) - (0 + 4\ln|-1| - 4\pi)\right. \\
&\quad \left. + (0 + 4\ln|-1| + 4\pi) - \left(-\sqrt{3} + 4\ln\left|-2-\sqrt{3}\right| + \frac{8\pi}{3}\right)\right] \\
&= \frac{1}{2}\left(2\sqrt{3} - \frac{16\pi}{3} + 8\pi\right) + 2\ln(2-\sqrt{3}) - 2\ln(2+\sqrt{3}) \\
&= \frac{4}{3}\pi + \sqrt{3} + 2\ln\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right) \\
&= \frac{4}{3}\pi + \sqrt{3} + 4\ln(2-\sqrt{3}).
\end{aligned}$$