2015.3 Question 2

1. Let $m = 1000$. Notice for all $n \geq m$,

$$n^2 = n \cdot n \geq m \cdot n \geq 1000n.$$

2. This statement is false. Let $s_n = (-1)^n$ and $t_n = -(-1)^n$. Then $s_n = 1$ and $t_n = -1$ for even ns, and $s_n = -1$ and $t_n = 1$ for odd ns.

   So $s_n \geq t_n$ for even ns, and $t_n \geq s_n$ for odd ns. Since there can be arbitrarily big even and odd numbers, neither of the statements are true for these sequences.

3. Let m_1 be the m for $(s_n) \leq (t_n)$ and m_2 be the m for $(t_n) \leq (u_n)$. Let $m = \max\{m_1, m_2\}$.

   Notice that for all $n \geq m$, we have $n \geq m_1$ and therefore $s_n leq t_n$, and $n \geq m_2$ amd therefore $t_n \leq u_n$.

   By the transitivity of the $\leq$ relation, we have therefore $s_n \leq u_n$, for all $n \geq m$. Therefore, this statement is true.

4. This statement is true. Let $m = 4$, we aim to prove that $2^n \geq n^2$ for all $n \geq m$.

   We first wish to prove the lemma: for all $n \geq 4$, we have $n^2 \geq 2n + 1$.

   This is equivalent to proving that $n^2 - 2n + 1 \geq 2$ for all $n \geq 4$.

   Notice that $n^2 - 2n + 1 = (n-1)^2 \geq (4-1)^2 = 9 \geq 2$ is true.

   This finishes our proof for the lemma.

   We show the original statement by mathematical induction.

   (a) **Base case.** For $n = 4$, we have $2^4 = 16 \geq 4^2 = 16$.

   (b) **Inductive step.** Assume that $2^k \geq k^2$ for some $n = k \geq 4$. We aim to show that $2^{k+1} \geq (k+1)^2$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2 \cdot k^2 \\ &= k^2 + k^2 \\ &\geq k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

   Therefore, by the principle of mathematical induction, we have $2^n \geq n^2$ for all $n \geq 4$, and this finishes our proof.