2015.3 Question 13

1. The cumulative distribution function of $X+Y$ at some value t is ratio of the area below the line $X+Y=t$ within the unit square $[0,1]^2$, against the area of the unit square (which is 1).

   When $0 \leq t \leq 1$, the area below is a triangle with vertices at $(0,0)$, $(0,t)$ and $(t,0)$. This means

$$F_{X+Y}(t) = \frac{1}{2}t^2.$$

   When $1 \leq t \leq 2$, the area below is the unit square subtracting the triangle with vertices at $(1,1)$, $(1,t-1)$ and $(t-1,1)$. This means

$$F_{X+Y}(t) = 1 - \frac{1}{2}[1-(t-1)]^2 = 1 - \frac{1}{2}(2-t)^2.$$

   Hence, we have

$$F_{X+Y}(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}t^2, & 0 \leq t < 1, \\ 1 - \frac{1}{2}(2-t)^2, & 1 \leq t < 2, \\ 1, & 2 \leq t. \end{cases}$$

2. Since $X+Y \in [0,2]$, $(X+Y)^{-1} \in \left[\frac{1}{2}, \infty\right)$. Let $t' \in \left[\frac{1}{2}, \infty\right)$, we have

$$\begin{aligned} F_{(X+Y)^{-1}}(t') &= \mathrm{P}\left(\frac{1}{X+Y} \leq t'\right) \\ &= \mathrm{P}\left(X+Y \geq \frac{1}{t'}\right) \\ &= 1 - \mathrm{P}\left(X+Y < \frac{1}{t'}\right) \\ &= 1 - F_{X+Y}\left(\frac{1}{t'}\right). \end{aligned}$$

   If $t' \in \left[\frac{1}{2}, 1\right]$, we have $t'^{-1} \in [1,2]$, and hence

$$\begin{aligned} F_{(X+Y)^{-1}}(t') &= 1 - F_{X+Y}\left(\frac{1}{t'}\right) \\ &= 1 - \left[1 - \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2\right] \\ &= \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2. \end{aligned}$$

   If $t' \in [1, \infty)$, we have $t'^{-1} \in (0,1]$, and hence

$$\begin{aligned} F_{(X+Y)^{-1}}(t') &= 1 - F_{X+Y}\left(\frac{1}{t'}\right) \\ &= 1 - \frac{1}{2}\left(\frac{1}{t'}\right)^2. \end{aligned}$$

   Therefore, the cumulative distribution function of $(X+Y)^{-1}$ is given by

$$F_{(X+Y)^{-1}}(t') = \begin{cases} 0, & t' < \frac{1}{2}, \\ \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2, & \frac{1}{2} \leq t' < 1, \\ 1 - \frac{1}{2}\left(\frac{1}{t'}\right)^2, & 1 \leq t'. \end{cases}$$

The probability density function of $(X+Y)^{-1}$ is

$$
\begin{aligned}
f_{(X+Y)^{-1}}(t') &= \frac{\mathrm{d}}{\mathrm{d}t'} F_{(X+Y)^{-1}}(t') \\
&= \begin{cases} \frac{1}{2} \cdot 2 \cdot \left(2 - \frac{1}{t'}\right) \cdot t'^{-2} = 2t'^{-2} - t'^{-3}, & \frac{1}{2} \leq t' < 1, \\ -(-2)\frac{1}{2}t'^{-3} = t'^{-3}, & 1 \leq t', \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

as desired.

The expectation of $\frac{1}{X+Y}$ is

$$
\begin{aligned}
\mathrm{E}\left(\frac{1}{X+Y}\right) &= \int_{\mathbb{R}} t f_{(X+Y)^{-1}}(t) \, \mathrm{d}t \\
&= \int_{\frac{1}{2}}^{1} t \cdot \left(2t^{-2} - t^{-3}\right) \mathrm{d}t + \int_{1}^{\infty} t \cdot t^{-3} \, \mathrm{d}t \\
&= \int_{\frac{1}{2}}^{1} \left(2t^{-1} - t^{-2}\right) \mathrm{d}t + \int_{1}^{\infty} t^{-2} \, \mathrm{d}t \\
&= \left[2 \ln t + t^{-1}\right]_{\frac{1}{2}}^{1} - \left[t^{-1}\right]_{1}^{\infty} \\
&= \left[\left(2 \ln 1 + 1^{-1}\right) - \left(2 \ln \frac{1}{2} + \left(\frac{1}{2}\right)^{-1}\right)\right] - (0 - 1) \\
&= [1 + 2 \ln 2 - 2] + 1 \\
&= 2 \ln 2.
\end{aligned}
$$

3. The cumulative distribution function of Y/X at some value t is the ratio of the area below the line $Y/X = t$ within the unit square $[0,1]^2$, against the area of the unit square.

   Since $0 \leq X, Y \leq 1$, we have $0 \leq Y/X < \infty$.

   When $0 \leq t \leq 1$, the area below is a triangle with vertices at $(0,0), (1,0)$ and $(1,t)$. Hence, we have

$$F_{Y/X}(t) = \frac{t}{2}.$$

   When $1 \leq t < \infty$, the area below is the whole unit square, subtracting the triangle with the vertices at $(0,0), (0,1)$ and $\left(1, \frac{1}{t}\right)$. Hence, we have

$$F_{Y/X}(t) = 1 - \frac{1}{2t}.$$

   Therefore,

$$F_{Y/X}(t) = \begin{cases} 0, & t < 0, \\ \frac{t}{2}, & 0 \leq t < 1, \\ 1 - \frac{1}{2t}, & 1 \leq t. \end{cases}$$

   Hence, we have for $0 < t' \leq 1$, we have

$$
\begin{aligned}
F_{\frac{X}{X+Y}}(t') &= \mathrm{P}\left(\frac{X}{X+Y} \leq t'\right) \\
&= \mathrm{P}\left(\frac{1}{t'} \leq \frac{X+Y}{X}\right) \\
&= \mathrm{P}\left(\frac{1}{t'} \leq 1 + \frac{Y}{X}\right) \\
&= \mathrm{P}\left(\frac{Y}{X} \geq \frac{1}{t'} - 1\right) \\
&= 1 - \mathrm{P}\left(\frac{Y}{X} \leq \frac{1}{t'} - 1\right) \\
&= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right).
\end{aligned}
$$

For $0 < t' \leq \frac{1}{2}$, we have $2 \leq \frac{1}{t'}$, and hence $1 \leq \frac{1}{t'} - 1$,

$$\begin{aligned}
F_{\frac{X}{X+Y}}(t') &= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right) \\
&= 1 - \left[1 - \frac{1}{2 \cdot \left(\frac{1}{t'} - 1\right)}\right] \\
&= \frac{1}{2 \cdot \left(\frac{1}{t'} - 1\right)} \\
&= \frac{t'}{2 - 2t'}.
\end{aligned}$$

For $\frac{1}{2} \leq t' \leq 1$, we have $1 \leq \frac{1}{t'} \leq 2$, and hence $0 \leq \frac{1}{t'} \leq 1$,

$$\begin{aligned}
F_{\frac{X}{X+Y}}(t') &= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right) \\
&= 1 - \frac{\frac{1}{t'} - 1}{2} \\
&= \frac{2 - \frac{1}{t'} + 1}{2} \\
&= \frac{3t' - 1}{2t'}.
\end{aligned}$$

Hence, we have

$$F_{\frac{X}{X+Y}}(t') = \begin{cases} 0, & t' \leq 0, \\ \frac{t'}{2-2t'}, & 0 < t' \leq \frac{1}{2}, \\ \frac{3t'-1}{2t'}, & \frac{1}{2} < t' \leq 1, \\ 1, & 1 < t'. \end{cases}$$

Differentiating gives

$$\begin{aligned}
f_{\frac{X}{X+Y}}(t') &= \frac{\mathrm{d}}{\mathrm{d}t'} F_{\frac{X}{X+Y}}(t') \\
&= \begin{cases} \frac{1 \cdot (2-2t') + 2t'}{(2-2t')^2} = \frac{1}{2(1-t')^2}, & 0 < t' \leq \frac{1}{2}, \\ \frac{3 \cdot 2t' - 2(3t'-1)}{4t'^2} = \frac{1}{2t'^2}, & \frac{1}{2} < t' \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

By symmetry, $\mathrm{E}\left(\frac{X}{X+Y}\right) = \mathrm{E}\left(\frac{Y}{X+Y}\right)$, but also

$$\mathrm{E}\left(\frac{X}{X+Y}\right) + \mathrm{E}\left(\frac{Y}{X+Y}\right) = \mathrm{E}\left(\frac{X}{X+Y} + \frac{Y}{X+Y}\right) = \mathrm{E}(1) = 1,$$

and hence

$$\mathrm{E}\left(\frac{X}{X+Y}\right) = \frac{1}{2}.$$

Using integration, we have

$$\begin{aligned}
\mathrm{E}\left(\frac{X}{X+Y}\right) &= \int_{\mathbb{R}} x f_{\frac{X}{X+Y}}(x)\,\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{x}{2(1-x)^2}\,\mathrm{d}x + \int_{\frac{1}{2}}^1 \frac{1}{2x}\,\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{x}{2}\,\mathrm{d}\frac{1}{1-x} + \frac{1}{2}\left[\ln x\right]_{\frac{1}{2}}^1 \\
&= \left[\frac{x}{2(1-x)}\right]_0^{\frac{1}{2}} - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{1-x}\,\mathrm{d}x + \frac{\ln 2}{2} \\
&= \frac{\frac{1}{2}}{2\cdot\frac{1}{2}} + \frac{1}{2}\left[\ln(1-x)\right]_0^{\frac{1}{2}} + \frac{\ln 2}{2} \\
&= \frac{1}{2} - \frac{\ln 2}{2} + \frac{\ln 2}{2} \\
&= \frac{1}{2}.
\end{aligned}$$