2015.3 Question 12

1. Let X be the random variable for the outcome of one die roll. It has probability distribution $\mathrm{P}(X = x) = \frac{1}{6}$ for $x = 1, 2, \ldots, 6$.

   Therefore, R_1 follows the probability distribution $\mathrm{P}(R_1 = x) = \frac{1}{6}$ fir $x = 0, 1, \ldots, 5$, since $R_1 = X \bmod 6$.

   This means that
$$G(x) = \frac{1}{6}\left(1 + x + x^2 + x^3 + x^4 + x^5\right).$$

   $R_2 = (X_1 + X_2) \bmod 6 = ((R_1)_a + (R_1)_b) \bmod 6$, and notice that,
$$G(x)^2 = \frac{1}{36}\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}\right).$$

   Therefore, combining the terms with the same powers modulo 6, we get
$$G_{R_2}(x) = \frac{1}{36}\left((1+5) + (2+4)x + (3+3)x^2 + (4+2)x^3 + (5+1)x^4 + 6x^5\right)$$

   which simplifies gives $G(x)$, as desired.

   Therefore, since $R_n = (X_1 + X_2 + \ldots + X_n) \bmod 6 = (R_{n-1} + R_1) \bmod 6$, by mathematical induction, we can conclude that the probability generating function for R_n is always $G(x)$.

   This means that the probability of R_n being a multiple of 6, is
$$\mathrm{P}\left(6 \mid R_n\right) = \frac{1}{6}.$$

2. Notice that $G_1(x)$, the probability generating function for T_1 must be
$$G_1(x) = \frac{1}{6}\left(1 + 2x + x^2 + x^3 + x^4\right).$$

   Therefore, notice that
$$G_1(x)^2 = \frac{1}{36}\left(1 + 4x + 6x^2 + 6x^3 + 7x^4 + 6x^5 + 3x^6 + 2x^7 + x^8\right),$$

   and combining the powers with the same remainder modulo 5, we have
$$G_2(x) = \frac{1}{36}\left(7 + 7x + 8x^2 + 7x^3 + 7x^4\right) = \frac{1}{36}\left(x^2 + 7y\right)$$

   where $y = 1 + x + x^2 + x^3 + x^4$, as desired.

   Expressing G_1 in terms of y, we have
$$G_1(x) = \frac{1}{6}(x + y).$$

   Experimenting with G_3, we notice
$$\begin{aligned} G_1(x) \cdot G_2(x) &= \frac{1}{6^3}(x + y)(x^2 + 7y) \\ &= \frac{1}{6^3}(x^3 + 7xy + x^2y + 7y^2). \end{aligned}$$

   But notice that up to the congruence of the powers modulo 5, we have $x^n y$ will simplify to simply y, and
$$(x + y)^2 = x^2 + y^2 + 2xy = x^2 + 7y$$
   from $G_1(x)^2 = G_2(x)$ implies that y^2 simplifies to $5y$.

   Therefore,
$$G_3(x) = \frac{1}{6^3}(x^3 + 7y + y + 7 \cdot 5y) = \frac{1}{6^3}(x^3 + 43y).$$

Now, we assert that

$$G_n(x) = \frac{1}{6^n}(x^{n \bmod 5} + \frac{6^n - 1}{5}y).$$

The base case is shown in G_1, and now we do the inductive step. Assume that

$$G_k(x) = \frac{1}{6^k}(x^{k \bmod 5} + \frac{6^k - 1}{5}y)$$

for some $k \in \mathbb{N}$.

$$\begin{aligned}
G_{k+1}(x) &= G_k(x) \cdot G_1(x) \\
&= \frac{1}{6^k} \cdot \left(x^{k \bmod 5} + \frac{6^k - 1}{5}y\right) \cdot \frac{1}{6} \cdot (x + y) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{k \bmod 5} \cdot x^1 + x^{k \bmod 5} \cdot y + x \cdot \frac{6^k - 1}{5}y + \frac{6^k - 1}{5}y^2\right) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{(k+1) \bmod 5} + y + \frac{6^k - 1}{5}y + \frac{6^k - 1}{5} \cdot 5y\right) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{(k+1) \bmod 5} + \left(\frac{6^k - 1}{5} + 6^k\right) y\right).
\end{aligned}$$

What remains to prove is that

$$\frac{6^k - 1}{5} + 6^k = \frac{6^{k+1} - 1}{5},$$

but this is straightforward since this is just trivial algebra.

So our assertion is true, and

$$G_n(x) = \frac{1}{6^n}(x^{n \bmod 5} + \frac{6^n - 1}{5}y).$$

Now, the probability of $5 \mid S_n$ is the coefficient of x^0 (the constant term) in $G_n(x)$.

If $5 \nmid n$, $x^{n \bmod 5}$ is not x^0, and therefore the only term that contributes to the constant term comes from y, therefore

$$\mathrm{P}\left(5 \mid S_n\right) = \frac{1}{6^n} \cdot \frac{6^n - 1}{5} = \frac{1}{5}\left(1 - \frac{1}{6^n}\right),$$

as required.

If $5 \mid n$, then $x^{n \bmod 5}$ will be $x^0 = 1$ contributing to the probability, hence

$$\mathrm{P}\left(5 \mid S_n\right) = \frac{1}{6^n} \cdot \left(1 + \frac{6^n - 1}{5}\right) = \frac{1}{5}\left(1 + \frac{4}{6^n}\right).$$