2015.3 Question 1

1. We have

$$\begin{aligned} I_n - I_{n+1} &= \int_0^\infty \frac{1}{(1+u^2)^n} \, \mathrm{d}x - \int_0^\infty \frac{1}{(1+u^2)^{n+1}} \, \mathrm{d}x \\ &= \int_0^\infty \frac{(1+u^2)-1}{(1+u^2)^{n+1}} \, \mathrm{d}x \\ &= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}} \, \mathrm{d}x. \end{aligned}$$

Notice that

$$\frac{\mathrm{d}(1+u^2)^{-n}}{\mathrm{d}x} = -\frac{2un}{(1+u^2)^{n+1}},$$

and therefore,

$$\frac{u \, \mathrm{d}x}{(1+u^2)^{n+1}} = -\frac{\mathrm{d}(1+u^2)^{-n}}{2n}.$$

Using integration by parts, we have

$$\begin{aligned} I_n - I_{n+1} &= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}} \, \mathrm{d}x \\ &= \int_0^\infty \left[-\frac{u \, \mathrm{d}(1+u^2)^{-n}}{2n} \right] \\ &= \frac{1}{2n} \left[\int_0^\infty \frac{\mathrm{d}u}{(1+u^2)^n} - \left[u \cdot (1+u^2)^{-n} \right]_0^\infty \right] \\ &= \frac{1}{2n} \left[I_n - (0-0) \right] \\ &= \frac{1}{2n} I_n, \end{aligned}$$

as desired.

Hence, we have

$$I_{n+1} = \left(1 - \frac{1}{2n} \right) I_n = \frac{2n-1}{2n} I_n.$$

Notice that

$$I_1 = \int_0^\infty \frac{\mathrm{d}u}{1+u^2} = [\arctan u]_0^\infty = \frac{\pi}{2},$$

and therefore, we have

$$\begin{aligned} I_{n+1} &= \frac{2n-1}{2n} I_n \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot I_{n-1} \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdot I_{n-2} \\ &\vdots \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{2 \cdot 1 - 1}{2 \cdot 1} \cdot I_1 \\ &= \frac{(2n-1)(2n-3)(2n-5) \cdots 3 \cdot 1}{(2n) \cdot (2n-2) \cdot (2n-4) \cdots 4 \cdot 2} \cdot \frac{\pi}{2}. \end{aligned}$$

Let

$$\begin{aligned} A &= (2n-1)(2n-3)(2n-5) \cdots 3 \cdot 1, \\ B &= (2n) \cdot (2n-2) \cdot (2n-4) \cdots 4 \cdot 2. \end{aligned}$$

Notice that

$$AB = (2n)!, B = 2^n \cdot n!,$$

and therefore

$$A = \frac{(2n)!}{2^n \cdot n!},$$

and hence

$$\begin{aligned} I_{n+1} &= \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1}{(2n) \cdot (2n-2) \cdot (2n-4) \cdots 4 \cdot 2} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!/(2^n \cdot n!)}{2^n \cdot n!} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!\pi}{2^{2n+1}(n!)^2}, \end{aligned}$$

as desired.

2. If we do the substitution $u = \frac{1}{x}$, we will have $u \to 0^+$ as $x \to \infty$, and $u \to \infty$ as $x \to 0^+$. We have $\mathrm{d}u = -\frac{1}{x^2}\,\mathrm{d}x$. Therefore,

$$\begin{aligned} J &= \int_0^\infty f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \int_\infty^0 f((u^{-1} - u)^2)\left(-x^2\,\mathrm{d}u\right) \\ &= \int_0^\infty u^{-2} f((u - u^{-1})^2)\,\mathrm{d}u, \end{aligned}$$

which is exactly the first equal sign as desired (since u is just an arbitrary variable).

For the second equal sign, notice that

$$\begin{aligned} 2J &= J + J \\ &= \int_0^\infty f((x - x^{-1})^2)\,\mathrm{d}x + \int_0^\infty x^{-2} f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x, \end{aligned}$$

and therefore

$$J = \frac{1}{2}\int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x.$$

For the final equal sign, consider the substitution $u = x - x^{-1}$. Note $\mathrm{d}u = \left(1 + x^{-2}\right)\mathrm{d}x$, and when $x \to 0^+$, $u \to -\infty$, when $x \to \infty$, $u \to \infty$. Therefore,

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^\infty f(u^2)\,\mathrm{d}u. \end{aligned}$$

Since $f(u^2) = f((-u)^2)$ for all $u \in \mathbb{R}$, we therefore have

$$\int_{-\infty}^0 f(u^2)\,\mathrm{d}u = \int_0^\infty f(u^2)\,\mathrm{d}u,$$

and hence

$$J = \int_0^\infty f(u^2)\,\mathrm{d}u,$$

as desired.

3. Notice that the integrand satisfies

$$\begin{aligned}\frac{x^{2n-2}}{(x^4-x^2+1)^n} &= \frac{1}{x^2}\cdot\frac{(x^2)^n}{(x^4-x^2+1)^n}\\ &= \frac{1}{x^2}\cdot\frac{1}{(x^2-1+x^{-2})^n}\\ &= \frac{1}{x^2}\cdot\frac{1}{[(x-x^{-1})^2+1]^n}.\end{aligned}$$

Therefore, consider the function $f_n(x)=\frac{1}{(x+1)^n}$, we have

$$\begin{aligned}\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}\,\mathrm{d}x &= \int_0^\infty \frac{1}{x^2}\cdot\frac{1}{[(x-x^{-1})^2+1]^n}\cdot\mathrm{d}x\\ &= \int_0^\infty x^{-2}f_n((x-x^{-1})^2)\,\mathrm{d}x\\ &= \int_0^\infty f_n(u^2)\,\mathrm{d}u\\ &= \int_0^\infty \frac{\mathrm{d}u}{(u^2+1)^n}\\ &= \frac{(2n-2)!\pi}{2^{2n-1}((n-1)!)^2}.\end{aligned}$$