2014.3 Question 8

Notice that there are $(k^{n+1}-1)-k^n+1=k^{n+1}-k^n=k^n(k-1)$ items in the summation. By the monotonic condition of the sequence in the question, we know that all the elements in the sum are greater than or equal to $f(k^n)$ and less than $f(k^{n+1})$. This immediately proves the inequality.

1. Let $k=2$. Since f is decreasing, we know that for all non-negative n, we have

   $$2^n\cdot(2-1)\cdot\frac{1}{2^{n+1}}\leq\sum_{r=2^n}^{2^{n+1}-1}\frac{1}{r}\leq 2^n\cdot(2-1)\cdot\frac{1}{2^n},$$

   which simplifies to

   $$\frac{1}{2}1\leq\sum_{r=2^n}^{2^{n+1}-1}\frac{1}{r}\leq 1.$$

   Summing this from $n=0$ to $n=N$ (which contains $(N+1)$ such inequalities) yields

   $$\frac{N+1}{2}\leq\sum_{r=1}^{2^{N+1-1}}\frac{1}{r}\leq N+1,$$

   as desired.

   We can show that this sum can be arbitrarily big by letting $N\to\infty$, and the lower bound of the sum $\frac{N+1}{2}\to\infty$. This means the infinite sum must diverge.

2. Let $k=2$. Since f is decreasing, we know that for all non-negative n, we have

   $$\sum_{r=2^n}^{2^{n+1}-1}\frac{1}{r^3}\leq 2^n\cdot(2-1)\cdot\frac{1}{(2^n)^3}=\frac{1}{2^{2n}}=\frac{1}{4^n}.$$

   Summing this from $n=0$ up to $n=N$ gives

   $$\sum_{r=1}^{2^{N+1}-1}\frac{1}{r^3}\leq\sum_{n=0}^{N}\frac{1}{4^n}=\frac{1-\frac{1}{4^N}}{1-\frac{1}{4}}=\frac{4}{3}\cdot\left(1-\frac{1}{4^{N+1}}\right).$$

   Let $N\to\infty$, the weak inequality remains. This gives

   $$\sum_{r=1}^{\infty}\frac{1}{r^3}\leq\frac{4}{3}\cdot 1=\frac{4}{3}$$

   as desired.

3. Using a probabilistic argument, from the set of three-digit non-negative integers (allowing leading-zeros) $\{0,1,2,\ldots,999\}$, each digit has a $\frac{1}{10}$ chance of being 2, and hence $\frac{9}{10}$ chance of not being 2. This means that the number of elements in this set not being 2 is equal to

   $$10^3\cdot\left(\frac{9}{10}\right)^n=9^3.$$

   But 0 is counted in the 9^3 as well, which is not included in $S(1000)$. Therefore, $S(1000)=9^3-1$.

   This method applies in general to n-digit numbers and for $S(10^n)=9^n-1$ as well.

   Let $f(i)$ be the i-th integer not having 2 in the decimal expansion in increasing order, and hence

   $$S(n)=\{f(i)\mid i\in\mathbb{N},f(i)<n\},$$

   and

   $$\sigma(n)=\sum_{i=1}^{S(n)}\frac{1}{f(i)}.$$

Let $k = 9$. Notice that $f(9^n) = f(S(10^n) + 1) = 10^n$ since 10^n is must be the next number satisfying the condition. Also, since f must be increasing on the integers, we have $x \mapsto \frac{1}{f(x)}$ is decreasing on the integers, and hence, for non-negative integers n

$$\sum_{r=9^n}^{9^{n+1}-1} \frac{1}{f(r)} \leq 9^n(9-1)\frac{1}{f(9^n)} = 8 \cdot \left(\frac{9}{10}\right)^n.$$

Summing this from $n = 0$ to $n = N$ gives

$$\sigma(10^{N+1}) = \sum_{r=0}^{9^{N+1}-1} \frac{1}{f(r)} \leq 8 \sum_{n=0}^{N} \left(\frac{9}{10}\right)^n = 80 \left[1 - \left(\frac{9}{10}\right)^{N+1}\right] < 80.$$

For all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $10^{N+1} \geq n$, and since σ is increasing, we must have $80 > \sigma(10^{N+1}) \geq \sigma(n)$, which finishes the proof.