2014.3 Question 7

1. Since P_1, P_2, P_3, P_4 are cyclic, they must satisfy that $\angle P_1P_2P_4 = \angle P_1P_3P_4$, which means $\angle P_1P_2Q = \angle QP_3P_4$. Also, we must have $\angle P_1QP_2 = \angle P_3QP_4$.

   This means that $\triangle P_1QP_2 \sim \angle P_4QP_3$ (in this order). Therefore, the ratio of the side lengths satisfy that
$$\frac{P_1Q}{QP_2} = \frac{P_4Q}{QP_3},$$
   and hence
$$P_1Q \cdot QP_3 = P_2Q \cdot QP_4$$
   as desired.

2. Since Q is the intersection of P_1P_3 and P_2P_4, Q is on P_1P_3, and hence the position vector of Q, $\mathbf{q}$ can be expressed as a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$, i.e.,
$$\mathbf{q} = b_1\mathbf{p}_1 + b_3\mathbf{p}_3$$
   where $b_1 + b_3 = 1$.

   Similarly,
$$\mathbf{q} = b_2\mathbf{p}_2 + b_4\mathbf{p}_4$$
   where $b_2 + b_4 = 1$.

   Hence
$$b_1\mathbf{p}_1 - b_2\mathbf{p}_2 + b_3\mathbf{p}_3 - b_4\mathbf{p}_4 = \mathbf{0}$$
   Let $a_1 = b_1, a_2 = -b_2, a_3 = b_3, a_4 = -b_4$, and we must have $\sum_{i=1}^{4} a_i = 0$, and $\sum_{i=1}^{4} a_i\mathbf{p}_i = \mathbf{0}$. Since $b_1 + b_3 = 1$ they must not be both zero, and hence a_1, a_2, a_3, a_4 are not all zero.

3. If we have $a_1 + a_3 = 0$, we must also have $a_2 + a_4 = 0$. Let $a_1 = \lambda, a_2 = \mu, a_3 = -\lambda, a_4 = -\mu$, we have
$$\lambda(\mathbf{p}_1 - \mathbf{p}_3) = \mu(\mathbf{p}_2 - \mathbf{p}_4).$$
   But since P_1P_3 and P_2P_4 intersect at one point, this means they must not be parallel, and hence one of λ and μ must be zero. But if one of them is zero the other one has to be as well, which means all of a_i are zero, which contradicts with given.

   Still, let $b_1 = a_1, b_2 = -a_2, b_3 = a_3, b_4 = -a_4$. From given, we must have $b_1 + b_3 = b_2 + b_4 = T$. By rearrangement of the given vector equation, we have
$$b_1\mathbf{p}_1 + b_3\mathbf{p}_3 = b_2\mathbf{p}_2 + b_4\mathbf{p}_4.$$
   If we divide both sides by T, we have
$$\frac{b_1}{b_1 + b_3}\mathbf{p}_1 + \frac{b_3}{b_1 + b_3}\mathbf{p}_3 = \frac{b_2}{b_2 + b_4}\mathbf{p}_2 + \frac{b_4}{b_2 + b_4}\mathbf{p}_4.$$
   The position vector represented on the left-hand side must be on the line P_1P_3, and on the right-hand side must be on the line P_2P_4. But they have a unique intersection at Q, which means both must represent the position vector of Q, which is exactly
$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}.$$
   It must be true that $a_3 : a_1 = P_1Q : QP_3$. This is because
$$\mathbf{q} = \mathbf{p}_1 + \frac{a_3}{a_1 + a_3}(\mathbf{p}_3 - \mathbf{p}_1).$$
   The magnitude of $\mathbf{p}_3 - \mathbf{p}_1$ is the length P_1P_3 and the distance Q has 'travelled' along P_1P_3 from P_1 is $\frac{a_3}{a_1+a_3}$ of the total.

This means

$$P_1Q = \frac{a_3}{a_1 + a_3} P_1P_3, P_3Q = \frac{a_1}{a_1 + a_3} P_1P_3.$$

Similarly,

$$P_2Q = \frac{a_4}{a_2 + a_4} P_2P_4, P_4Q = \frac{a_2}{a_2 + a_4} P_2P_4.$$

From the first part of the question we have

$$\frac{a_1a_3}{(a_1 + a_3)^2}(P_1P_3)^2 = \frac{a_2a_4}{(a_2 + a_4)^2}(P_2P_4)^2.$$

But since $a_1 + a_2 + a_3 + a_4 = 0$, $a_1 + a_3 = -a_2 - a_4$, and hence $(a_1 + a_3)^2 = (a_2 + a_4)^2$. This means

$$a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2,$$

as desired.