2014.3 Question 6

Since $f''(t) > 0$ for $t \in (0, x_0)$, we must have that for all $x \in (0, x_0)$, we have $f''(t) > 0$ for $t \in (0, x)$, and hence

$$\int_0^x f''(t)\,\mathrm{d}t = f'(x) - f'(0) > 0.$$

But since $f'(0) = 0$, this implies that $f'(x) > 0$ for $x \in (0, x_0)$.

Repeating this exact step gives that $f(x) > 0$ for $x \in (0, x_0)$ as desired.

1. We would like to show $f(x) = 1 - \cos x \cosh x > 0$ for $x \in \left(0, \frac{1}{2}\pi\right)$. Notice that $f(0) = 1 - 1 \cdot 1 = 0$, and

   $$f'(x) = \sin x \cosh x - \cos x \sinh x,$$

   which means

   $$f'(0) = 0 \cdot 1 - 1 \cdot 0 = 0.$$

   Further differentiation gives

   $$f''(x) = \cos x \cosh x + \sin x \sinh x + \sin x \sinh x - \cos x \cosh x = 2 \sin x \sinh x.$$

   If $x \in \left(0, \frac{\pi}{2}\right)$, we have $\sin x > 0$ and $\sinh x > 0$, which gives $f''(x) > 0$.

   From the lemma we proved we have $f(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, which is exactly $\cos x \cosh x < 1$ as desired.

2. What is desired is to show $\sin x \cosh x - x > 0$ and $x^2 - \sin x \sinh x > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$.

   Let $g(x) = \sin x \cosh x - x$ and $h(x) = x^2 - \sin x \sinh x$. $g(0) = 0 \cdot 1 - 0 = 0$ and $h(0) = 0^2 - 0 \cdot 0 = 0$.

   Differentiating gives

   $$g'(x) = \cos x \cosh x + \sin x \sinh x - 1,$$

   and

   $$h'(x) = 2x - \cos x \sinh x - \sin x \cosh x.$$

   Hence,

   $$g'(0) = 1 \cdot 1 + 0 \cdot 0 - 1 = 0,$$

   and

   $$h'(0) = 2 \cdot 0 - 1 \cdot 0 - 0 \cdot 1 = 0.$$

   Differentiating this again gives

   $$g''(x) = -\sin x \cosh x + \cos x \sinh x + \cos x \sinh x + \sin x \cosh x = 2 \cos x \sinh x,$$

   and

   $$h''(x) = 2 + \sin x \sinh x - \cos x \cosh x - \cos x \cosh x - \sin x \sinh x = 2 - 2 \cos x \cosh x.$$

   For $x \in (0, \frac{\pi}{2})$, we notice that $\cos x > 0$ and $\sinh x > 0$, and so $g''(x) > 0$. Also, notice that $h''(x) = 2f(x)$ so $h''(x) > 0$.

   Hence, $g(x) > 0, h(x) > 0$ when $x \in (0, \frac{\pi}{2})$ which proves the result as desired.