2014.3 Question 5

$ABCD$ is a parallelogram if and only if AB is parallel and equal to DC. This is true if and only if,

$$\overrightarrow{AB} = \overrightarrow{DC},$$

and using complex representation (which is also equivalent)

$$b - a = c - d.$$

This is equivalent to

$$a + c = b + d$$

so we are done.

In this case, $ABCD$ is further a square if and only if it is both a rhombus and a rectangle. It is a rhombus if and only if the two diagonals, AC and BD, are perpendicular to each other, and a rectangle if and only if the two diagonals, AC and BD, have equal length.

This is equivalent to $\overrightarrow{BD}$ being $\overrightarrow{AC}$ rotated 90 degrees anti-clockwise exactly (due to the labelling as defined), and using complex representation (which is equivalent)

$$i(c - a) = (d - b).$$

Flipping the signs on both sides (which is reversible) gives

$$i(a - c) = (b - d)$$

as desired.

1. X is the centre of the square constructed externally along the edge PQ if and only if $\overrightarrow{PX}$ is $\overrightarrow{PQ}$ rotated clockwise by 45 degrees and scaled down by a factor of $\sqrt{2}$. In complex notation, this is equivalent to

   $$x - p = (q - p) \cdot \frac{1}{\sqrt{2}} \cdot e^{-i\frac{\pi}{4}}.$$

   But notice that $e^{-i\frac{\pi}{4}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1 - i)$, and hence this equation is equivalent to

   $$x = \frac{1}{2}(q - p)(1 - i) + p = \frac{(1 + i)p + (1 - i)q}{2},$$

   as desired.

2. Similarly, we have

   $$\begin{aligned} y &= \frac{(1 + i)q + (1 - i)r}{2}, \\ z &= \frac{(1 + i)r + (1 - i)s}{2}, \\ t &= \frac{(1 + i)s + (1 - i)t}{2}. \end{aligned}$$

   $XYZT$ is a square, if and only if

   $$x + z = y + t$$

   and

   $$i(x - z) = y - t.$$

   For the first one, this is equivalent to

   $$(1 + i)p + (1 - i)q + (1 + i)r + (1 - i)s = (1 - i)p + (1 + i)q + (1 - i)r + (1 + i)s,$$

   which is equivalent to

   $$p + r = q + s,$$

   which is equivalent to $PQRS$ being a parallelogram.

For the second one, this is equivalent to

$$i \cdot ((1+i)p + (1-i)q - (1+i)r - (1-i)s) = -(1-i)p + (1+i)q + (1-i)r - (1+i)s,$$

which is equivalent to

$$-(1+i)p + (1+i)q + (1-i)r - (1+i)s = -(1-i)p + (1+i)q + (1-i)r - (1+i)s,$$

which is trivially true.

This shows that $XYZT$ being square is equivalent to $PQRS$ being a parallelogram as desired.