2014.3 Question 4

1. We have

$$\begin{aligned}I - I_0 &= \int_0^1 \left[(y')^2 - y^2 - (y' + y\tan x)^2\right] \mathrm{d}x \\ &= -\int_0^1 \left[y^2 + y^2\tan^2 x + 2yy'\tan x\right] \mathrm{d}x \\ &= -\int_0^1 \left[y^2(1+\tan^2 x) + 2yy'\tan x\right] \mathrm{d}x \\ &= -\int_0^1 \left(y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x\right) \mathrm{d}x.\end{aligned}$$

But notice that

$$\frac{\mathrm{d}}{\mathrm{d}x} y^2 \tan x = y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x,$$

and hence

$$\begin{aligned}I - I_0 &= -\int_0^1 \left(y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x\right) \mathrm{d}x \\ &= -\left[y^2 \tan x\right]_0^1 \\ &= -(y(1)^2 \tan 1 - 0^2 \tan 0) \\ &= -(0^2 \tan 1 - 0) \\ &= 0,\end{aligned}$$

as desired.

This gives $I = I_1$. Also, notice that the integrand of I_1 is $(y' + y\tan x)^2$ is always non-negative, which means $I_1 \geq 0$, taking 0 only when $y' + y\tan x = 0$ for all $x \in (0, 1)$.

$$\begin{aligned}y' + y\tan x &= 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -y\tan x \\ \frac{\mathrm{d}y}{y} &= -\tan x \,\mathrm{d}x \\ \ln|y| &= -\ln|\sec x| + C \\ y &= A\cos x.\end{aligned}$$

When $x = 1$, $y = 0$, hence $A = 0$ since $\cos 1 \neq 0$. This means $I_1 = 0$ if and only if $y = 0$ for all $x \in [0, 1]$.

Since $I = I_1$, we know that $I \geq 0$, with the equal sign holding if and only if $y = 0$ for all $x \in [0, 1]$.

2. Let

$$J_0 = \int_0^1 (y' + ay\tan bx)^2 \,\mathrm{d}x,$$

and we have

$$\begin{aligned}J - J_0 &= \int_0^1 \left[((y')^2 - a^2y^2) - (y' + ay\tan bx)^2\right] \mathrm{d}x \\ &= -\int_0^1 \left[a^2y^2 + a^2y^2\tan^2 bx + 2y' \cdot y \cdot a \cdot \tan bx\right] \mathrm{d}x \\ &= -\int_0^1 \left[a^2y^2\sec^2 bx + 2y' \cdot y \cdot a \cdot \tan bx\right] \mathrm{d}x \\ &= -a\int_0^1 \left[ay^2\sec^2 bx + 2y' \cdot y \cdot \tan bx\right] \mathrm{d}x.\end{aligned}$$

Notice that if we let $b = a$, we have

$$\frac{\mathrm{d}y^2 \tan bx}{\mathrm{d}x} = 2yy' \tan bx + by^2 \sec^2 bx = 2yy' \tan bx + ay^2 \sec^2 bx.$$

This means

$$\begin{aligned} J - J_0 &= -a \int_0^1 \left[ay^2 \sec^2 bx + 2y' \cdot y \cdot \tan bx\right] \mathrm{d}x \\ &= -a \left[y^2 \tan ax\right]_0^1 \\ &= -a(y(1)^2 \tan a - 0^2 \tan 0) \\ &= 0. \end{aligned}$$

This means $J = J_0$.

Since the integrand of J_0 is a square, we know $J_0 \geq 0$ and hence $J \geq 0$.

This is only valid when $ax < \frac{\pi}{2}$ for $x \in [0, 1]$ (since otherwise this range will cross an undefined point), which means $a < \frac{\pi}{2}$.

When $a = \frac{\pi}{2}$, consider $y = \cos ax$. Notice that $y' = -a \sin ax$, and therefore

$$\begin{aligned} J &= \int_0^1 ((-a \sin ax)^2 - a^2 \cos^2 ax) \,\mathrm{d}x \\ &= -a^2 \int_0^1 (\cos^2 ax - \sin^2 ax) \,\mathrm{d}x \\ &= -a^2 \int_0^1 \cos(2ax) \,\mathrm{d}x \\ &= -a^2 \left[\frac{\sin 2ax}{2a}\right]_0^1 \\ &= -\frac{a}{2} \left[\sin \pi x\right]_0^1 \\ &= -\frac{a}{2}(0 - 0) \\ &= 0, \end{aligned}$$

but y is not uniformly zero.