2014.3 Question 3

1. Consider the point on the curve whose gradient is equal to m. Since on the curve, $x = \frac{y^2}{4a}$, and hence
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2a} = \frac{1}{m},$$
which solves to $y_0 = \frac{2a}{m}$, and hence $x_0 = \frac{a}{m^2}$, the tangent to this point is $y = mx + \frac{a}{m}$.

   If $\frac{a}{m} < c$ and $mc > a$, this means that the line $y = mx + c$ is above the tangent. Let $\theta = \arctan m$, and we know the perpendicular distance between these lines will be
$$\left(c - \frac{a}{m}\right) \cdot \cos\theta = \left(c - \frac{a}{m}\right) \cdot \frac{1}{\sqrt{m^2+1}} = \frac{cm-a}{m\sqrt{m^2+1}}.$$

   If $\frac{a}{m} \geq c$ and $mc \leq a$, this means that the line $y = mx + c$ is the tangent (in the equal case) or below the tangent (in the less-than case), which both means the line $y = mx + c$ intersects with the parabola.

   Hence, when $mc \leq a$, the shortest distance is always 0.

2. The distance d between $(p, 0)$ and $(at^2, 2at)$ can be expressed as
$$\begin{aligned} d^2 &= (at^2 - p)^2 + (2at)^2 \\ &= a^2t^4 - 2apt^2 + p^2 + 4a^2t^2 \\ &= a^2t^4 + 2a(2a-p)t^2 + p^2. \end{aligned}$$

   We would like to minimise $d \geq 0$, which is the same as minimising d^2.

   The minimum of the quadratic function
$$f(x) = a^2x^2 + 2a(2a-p)x + p^2$$
occurs when
$$x = -\frac{2a(2a-p)}{2 \cdot a^2} = \frac{p-2a}{a} = \frac{p}{a} - 2.$$

   However, $d^2 = f(t^2)$ and t^2 can only be non-negative.

   If $\frac{p}{a} - 2 \geq 0$, $\frac{p}{a} \geq 2$, then this value can be taken, and the minimum will be
$$d^2 = \frac{4a^2p^2 - [2a(2a-p)]^2}{4a^2} = p^2 - (2a-p)^2 = -4a^2 + 4ap = 4a(p-a)$$
and the minimal d will be
$$d = 2\sqrt{a(p-a)}.$$

   In the other case where $\frac{p}{a} < 2$, to let the t^2 value to be as close as possible to the symmetric axis, we would like $t^2 = 0$, at which point the minimal distance will be
$$d^2 = f(0) = p^2,$$
and the minimal d will be
$$d = p.$$

   The circle described is simply a circle centred at $(p, 0)$ with radius b. Therefore, the shortest distance will be $d - b$ if $d > b$, and 0 otherwise.

   To put this into cases,

   - If $p \geq 2a$, $d = 2\sqrt{a(p-a)}$.
     - If $2\sqrt{a(p-a)} > b$, i.e. $b^2 < 4a(p-a)$, the shortest distance is $2\sqrt{a(p-a)} - b$.
     - Otherwise, $2\sqrt{a(p-a)} \leq b$, i.e. $b^2 \geq 4a(p-a)$, the shortest distance is 0.
   - Otherwise, $p < 2a$, $d = p$.
     - If $p > b$, the shortest distance is $p - b$.
     - Otherwise, $p \leq b$, the shortest distance is 0.