2014.3 Question 2

1. Since $u = \cosh x$, $\cosh 2x = 2\cosh^2 x - 1 = 2u^2 - 1$, and $\sinh x \, \mathrm{d}x = \mathrm{d}\cosh x = \mathrm{d}u$. Hence,

$$\begin{aligned}\int \frac{\sinh x}{\cosh 2x} \, \mathrm{d}x &= \int \frac{\mathrm{d}u}{2u^2 - 1} \\ &= \int \frac{1}{2} \left(\frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1} \right) \mathrm{d}u \\ &= \frac{1}{2\sqrt{2}} \left(\ln\left|\sqrt{2}u - 1\right| - \ln\left|\sqrt{2}u + 1\right| \right) + C \\ &= \frac{1}{2\sqrt{2}} \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C,\end{aligned}$$

as desired.

2. Let $u = \sinh x$, $\cosh 2x = 1 + 2\sinh^2 x = 1 + 2u^2$, and $\cosh x \, \mathrm{d}x = \mathrm{d}\sinh x = \mathrm{d}u$. Hence,

$$\begin{aligned}\int \frac{\cosh x}{\cosh 2x} \, \mathrm{d}x &= \int \frac{\mathrm{d}u}{1 + 2u^2} \\ &= \frac{1}{\sqrt{2}} \arctan()\sqrt{2}u) + C \\ &= \frac{1}{\sqrt{2}} \arctan()\sqrt{2}\sinh x) + C.\end{aligned}$$

3. Notice that

$$\frac{\cosh x}{\cosh 2x} - \frac{\sinh x}{\cosh 2x} = \frac{2e^{-x}}{e^{2x} + e^{-2x}} = \frac{2e^x}{1 + e^{4x}}.$$

Let $u = e^x$, $\mathrm{d}u = \mathrm{d}e^x = e^x \, \mathrm{d}x$, and therefore

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}u}{1 + u^4} &= \int_{-\infty}^0 \frac{e^x \, \mathrm{d}x}{1 + x^4} \\ &= \frac{1}{2} \int_{-\infty}^0 \frac{\cosh x}{\cosh 2x} - \frac{\sinh x}{\cosh 2x} \, \mathrm{d}x \\ &= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \arctan(\sqrt{2}\sinh x) - \frac{1}{2\sqrt{2}} \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| \right]_{-\infty}^0 \\ &= \frac{1}{4\sqrt{2}} \left[2\arctan(\sqrt{2}\sinh x) - \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| \right]_{-\infty}^0 \\ &= \frac{1}{4\sqrt{2}} \left[\left(0 - \ln\left| \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right| \right) - \left(2 \cdot (-\frac{\pi}{2}) - \ln|1| \right) \right] \\ &= \frac{1}{4\sqrt{2}} \left[\pi - 2\ln(\sqrt{2} - 1) \right] \\ &= \frac{\pi + 2\ln(\sqrt{2} + 1)}{4\sqrt{2}},\end{aligned}$$

as desired.