2014.3 Question 13

1. Let this condition be C_1. Since the game ends in the first round, the score must remain to be zero, and therefore
$$P(N = 0 \mid C_1) = 1,$$
and for all other $n \in \mathbb{N}$ where $n \neq 0$,
$$P(N = n \mid C_1) = 0.$$
This means the p.g.f. for N conditional under C_1 is just simply $G(t \mid C_1) = P(N = 0 \mid C_1) \cdot t^0 = 1$.

2. Denote this condition be C_2. Since in the first round, the game score does not change, and after the first round it is just as if this was a new game, so for all $n \in \mathbb{N} \cup \{0\}$, we must have
$$P(N = n \mid C_2) = P(N = n),$$
and hence
$$G(t \mid C_2) = \sum_{n=0}^{\infty} P(N = n \mid C_2) \cdot x^n = \sum_{n=0}^{\infty} P(N = n) \cdot t^n = G(t).$$

3. Denote the condition where the score is increased by 1 as C_3. Since in the first round the game score increased by one, and after the first round it is just as if this was a new game, so for all $n \in \mathbb{N}$, we must have
$$P(N = n \mid C_3) = P(N = n - 1),$$
and
$$P(N = 0 \mid C_3) = 0.$$
Hence,
$$G(t \mid C_3) = \sum_{n=0}^{\infty} P(N = n \mid C_3) \cdot x^n = \sum_{n=1}^{\infty} P(N = n - 1) \cdot t^n = t \cdot \sum_{n=0}^{\infty} P(N = n) \cdot t^n = tG(t).$$
Since in the first round, one of C_1, C_2 and C_3 must happen, we must have that
$$G(t) = P(C_1) \cdot G(t \mid C_1) + P(C_2) \cdot G(t \mid C_2) + P(C_3) \cdot G(t \mid C_3) = a + bG(t) + ctG(t).$$
Hence, rearranging gives
$$(1 - b - ct)G(t) = a,$$
and hence
$$G(t) = \frac{a}{(1 - b) - ct} = \frac{a/(1 - b)}{1 - ct/(1 - b)}$$
Hence, using the infinite expansion, we have
$$\begin{aligned} G(t) &= \frac{a}{1 - b} \cdot \sum_{k=0}^{\infty} \left(\frac{ct}{1 - b}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{a}{1 - b} \cdot \frac{c^k}{(1 - b)^k} \cdot t^k \\ &= \sum_{k=0}^{\infty} \frac{ac^k}{(1 - b)^{k+1}} \cdot t^k. \end{aligned}$$
But the coefficient before t^n is precisely the probability $P(N = n)$. This means
$$P(N = n) = \frac{ac^k}{(1 - b)^{k+1}},$$
as desired.

4. We know that $\mu = G'(1)$. We can find that

$$G'(t) = \frac{ac}{[(1-b)-ct]^2},$$

and evaluating this at $t = 1$ gives

$$\mu = G'(1) = \frac{ac}{(1-b-c)^2} = \frac{ac}{a^2} = \frac{c}{a}.$$

Therefore, we have $c = \mu a$

$$\begin{aligned} \mathrm{P}(N=n) &= \frac{ac^k}{(a+c)^{k+1}} \\ &= \frac{a(\mu a)^k}{(a+\mu a)^{k+1}} \\ &= \frac{a\mu^k a^k}{a^{k+1}(1+\mu)^{k+1}} \\ &= \frac{\mu^k}{\mu^{k+1}}, \end{aligned}$$

as desired.