2014.3 Question 12

1. Notice that x_m is such that
$$\mathrm{P}(X \leq x_m) = F(x_m) = \frac{1}{2}.$$
y_m is such that
$$\mathrm{P}(Y \leq y_m) = \mathrm{P}(e^X \leq y_m) = \mathrm{P}(X \leq \ln y_m) = F(\ln y_m) = \frac{1}{2}.$$
Therefore,
$$F(x_m) = F(\ln y_m) = \frac{1}{2}.$$
Therefore, $x_m = \ln y_m$, and $y_m = e^{x_m}$.

2. Notice that the cumulative distribution function $G(y)$ of Y satisfies that
$$G(y) = \mathrm{P}(Y \leq y) = \mathrm{P}(e^X \leq y) = \mathrm{P}(X \leq \ln y) = F(\ln y).$$
Therefore, differentiating both sides w.r.t. y gives that the probability density function of Y, $g(y)$ satisfies
$$g(y) = \frac{1}{y} f(\ln y),$$
as desired.

   The mode of Y, λ must satisfy that $g'(\lambda) = 0$. By quotient rule, we have
$$g'(y) = \frac{f'(\ln y) \cdot \frac{1}{y} \cdot y - 1 \cdot f(\ln y)}{y^2} = \frac{f'(\ln y) - f(\ln y)}{y^2}.$$
Therefore, $g'(\lambda) = 0$ implies that $f'(\ln \lambda) = f(\ln \lambda)$ as desired.

3. This is because it is simply a horizontal shift of $f(x)$ in the positive x direction by σ^2 (i.e. this is the integral of $f(x - \sigma^2)$), and this improper integral on $\mathbb{R}$ will evaluate to the same value as integrating $f(x)$, which is simply 1.

   Expanding the exponent of the integrand gives
$$\begin{aligned} -\frac{(x - \mu - \sigma^2)^2}{2\sigma^2} &= -\frac{(x-\mu)^2 + \sigma^4 - 2\sigma^2(x-\mu)}{2\sigma^2} \\ &= -\frac{(x - \mu^2)}{2\sigma^2} - \frac{1}{2}\sigma^2 + (x - \mu). \end{aligned}$$
Hence,
$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(e^x) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x \cdot e^{-(x-\mu)^2/(2\sigma^2)} \, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)+x} \, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\mu+\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)+x-\frac{1}{2}\sigma^2-\mu} \, \mathrm{d}x \\ &= e^{\mu+\frac{1}{2}\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma)^2/(2\sigma^2)} \, \mathrm{d}x \\ &= e^{\mu+\frac{1}{2}\sigma^2}, \end{aligned}$$
as desired.

4. When $X \sim \mathrm{N}(\mu, \sigma^2)$, $x_m = \mu$ and therefore $y_m = e^\mu$. Differentiating the p.d.f. for X gives

$$\begin{aligned} f'(x) &= \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{-2(x-\mu)}{2\sigma^2} \cdot e^{-(x-\mu)^2/(2\sigma^2)} \\ &= -\frac{x-\mu}{\sigma^2 \cdot \sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/(2\sigma^2)}. \end{aligned}$$

Therefore, $f(x) = f'(x)$ when $-\frac{x-\mu}{\sigma^2} = 1$. This is precisely when $x = \mu - \sigma^2$, which means

$$\lambda = e^{\mu-\sigma^2}.$$

Now, since $\mathrm{E}(Y) = e^{\mu+\frac{1}{2}\sigma^2}, y_m = e^\mu, \lambda = e^{\mu-\sigma^2}$, and $\sigma \neq 0$ so $\sigma^2 > 0$, this gives the result

$$\lambda < y_m < \mathrm{E}(Y)$$

as desired.