2014.3 Question 1

Notice that

$$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+ac+bc)x^2 + abcx^3,$$

and by comparing coefficients we have

$$q = bc + ca + ab, r = abc$$

1. Using the identities for the logarithms, we have

$$\begin{aligned}\ln(1+qx^2+rx^3) &= \ln(1+ax) + \ln(1+bx) + \ln(1+cx)\\ &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(ax)^k}{k} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(bx)^k}{k} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(cx)^k}{k}\\ &= \sum_{k=1}^{\infty}(-1)^{k+1}x^k\frac{a^k+b^k+c^k}{k},\end{aligned}$$

   and hence

$$S_k = \frac{a^k+b^k+c^k}{k},$$

   as desired.

2. Since

$$\begin{aligned}S_2 &= \frac{a^2+b^2+c^2}{2}\\ &= \frac{(a+b+c)^2 - 2(ab+bc+ca)}{2}\\ &= \frac{0^2-2q}{2}\\ &= -q,\end{aligned}$$

$$\begin{aligned}S_3 &= \frac{a^3+b^3+c^3}{3}\\ &= \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc}{3}\\ &= abc\\ &= r,\end{aligned}$$

   and

$$\begin{aligned}S_5 &= \frac{a^5+b^5+c^5}{5}\\ &= \frac{(a^2+b^2+c^2)(a^3+b^3+c^3) - a^2b^2(a+b) - a^2c^2(a+c) - b^2c^2(b+c)}{5}\\ &= \frac{(-2q)(3r) + a^2b^2c + b^2c^2a + a^2c^2b}{5}\\ &= \frac{-6qr + abc(ab+bc+ac)}{5}\\ &= \frac{-6qr+qr}{5}\\ &= -qr.\end{aligned}$$

   Therefore, $S_2S_3 = S_5$ as desired.

3. Notice that

$$\begin{aligned}
S_7 &= \frac{a^7+b^7+c^7}{7} \\
&= \frac{(a^2+b^2+c^2)(a^5+b^5+c^5)}{7} \\
&= \frac{(-2q)\cdot(-5qr)-a^2b^2(a^3+b^3)-b^2c^2(b^3+c^3)-a^2c^2(a^3+c^3)}{7} \\
&= \frac{10q^2r-a^2b^2(3r-c^3)-b^2c^2(3r-a^3)-a^2c^2(3r-b^3)}{7} \\
&= \frac{10q^2r-3r(a^2b^2+b^2c^2+a^2c^2)+a^2b^2c^2(a+b+c)}{7} \\
&= \frac{10q^2r-3r\left[(ab+bc+ac)^2-2abc(a+b+c)\right]+r^2\cdot 0}{7} \\
&= \frac{10q^2r-3q^2r}{7} \\
&= q^2r.
\end{aligned}$$

Also, $S_2S_5=(-q)\cdot(-qr)=q^2r$, so $S_2S_5=S_7$ as desired.

4. Let $a=1, b=1, c=-2$. $q=bc+ca+ab=-3, r=-2$. This means $S_2=-q=3, S_7=q^2r=-18$. Notice that

$$S_9=\frac{a^9+b^9+c^9}{7}=\frac{1^9+1^9+(-2)^9}{9}=-\frac{510}{9}=-\frac{170}{3},$$

and this is obviously not S_2S_7 which gives a counterexample and the original statement is not true.