2013.3 Question 8

By the formula of the sum for a geometric series, we have

$$\begin{aligned}\sum_{r=0}^{n-1} \exp(2i(\alpha + r\pi/n)) &= \exp(2i(\alpha + 0\pi/n)) \cdot \frac{1 - \exp(2i\pi/n)^n}{1 - \exp(2i\pi/n)} \\ &= \exp(2i\alpha) \cdot \frac{1 - \exp(2i\pi)}{1 - \exp(2i\pi/n)} \\ &= \exp(2i\alpha) \cdot \frac{1 - 1}{1 - \exp(2i\pi/n)} \\ &= 0,\end{aligned}$$

since the denominator is not 0.

By geometry, we have

$$r \cos\theta + s = d,$$

and hence

$$s = d - r\cos\theta.$$

Since $r = ks = k(d - r\cos\theta)$, we have

$$r = \frac{kd}{1 + k\cos\theta}.$$

Let L_1 be an angle α to horizontal, then L_j is angle $\alpha + (j-1)\pi/n$ angle to the horizontal for $j = 1, 2, \ldots, n$. Let $\theta_j = \alpha + (j-1)\pi/n$, and we have

$$\begin{aligned}l_j &= r|_{\theta=\theta_j} + r|_{\theta=\theta_j+\pi} \\ &= kd\left(\frac{1}{1 + k\cos\theta_j} + \frac{1}{1 + k\cos(\theta_j + \pi)}\right) \\ &= kd\left(\frac{1}{1 + k\cos\theta_j} + \frac{1}{1 - k\cos\theta_j}\right) \\ &= kd \cdot \frac{1 + k\cos\theta_j + 1 - k\cos\theta_j}{1 - k^2\cos^2\theta_j} \\ &= \frac{2kd}{1 - k^2\cos^2\theta_j}.\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{l_j} &= \frac{1}{2kd} \sum_{j=1}^{n} (1 - k^2 \cos^2 \theta_j) \\
&= \frac{1}{2kd} \left[n - k^2 \sum_{j=1}^{n} \cos^2 \left(\alpha + (j-1)\pi/n \right) \right] \\
&= \frac{1}{2kd} \left[n - \frac{k^2}{2} \cdot \sum_{j=1}^{n} \left[1 + \cos 2 \left(\alpha + (j-1)\pi/n \right) \right] \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{j=1}^{n} \cos 2 \left(\alpha + (j-1)\pi/n \right) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{r=0}^{n-1} \cos 2 \left(\alpha + r\pi/n \right) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot \sum_{r=0}^{n-1} \operatorname{Re} \exp(2i \left(\alpha + r\pi/n \right)) \right] \\
&= \frac{1}{2kd} \left[n - \frac{nk^2}{2} - \frac{k^2}{2} \cdot 0 \right] \\
&= \frac{1}{2kd} \cdot \frac{n(2-k^2)}{2} \\
&= \frac{n(2-k^2)}{4kd},
\end{aligned}$$

as desired.