2013.3 Question 7

1. We notice that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y^3\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3\right) \\ &= 0,\end{aligned}$$

and so E must be constant.

So hence

$$\begin{aligned}E(x) &= E(0) \\ &= 0^2 + \frac{1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

Therefore,

$$y^4 = 2\left[E(x) - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] \leq 2E(x) = 1,$$

and hence

$$|y(x)| \leq 1.$$

2. We notice that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= 2 \cdot \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2v}{\mathrm{d}x^2} + 2\sinh v\frac{\mathrm{d}v}{\mathrm{d}x} \\ &= 2\frac{\mathrm{d}v}{\mathrm{d}x} \cdot \left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \sinh v\right) \\ &= 2\frac{\mathrm{d}v}{\mathrm{d}x} \cdot \left(-x\frac{\mathrm{d}v}{\mathrm{d}x}\right) \\ &= -2x\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2,\end{aligned}$$

so when $x \geq 0$, since $\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \geq 0$, we must have

$$\frac{\mathrm{d}E}{\mathrm{d}x} \leq 0.$$

Therefore, for $x \geq 0$, $E(x) \leq E(0) = 0^2 + 2\cosh\ln 3 = 3 + \frac{1}{3} = \frac{10}{3}$. Hence,

$$\begin{aligned}\cosh v(x) &= \frac{E(x) - \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2}{2} \\ &\leq \frac{\frac{10}{3}}{2} \\ &= \frac{5}{3}.\end{aligned}$$

3. Notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 &= 2 \cdot \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}^2v}{\mathrm{d}x^2} \\ &= -2 \cdot \frac{\mathrm{d}w}{\mathrm{d}x} \cdot \left[(5\cosh x - 4\sinh x - 3) \cdot \frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w)\right].\end{aligned}$$

We also notice that

$$\begin{aligned}\int (w\cosh w + 2\sinh w)\,\mathrm{d}w &= \int w\cosh w\,\mathrm{d}w + 2\cosh w \\ &= \int w\,\mathrm{d}\sinh w + 2\cosh w + C \\ &= w\sinh w - \int \sinh w\,\mathrm{d}w + 2\cosh w + C \\ &= w\sinh w - \cosh w + 2\cosh w + C \\ &= w\sinh w + \cosh w + C,\end{aligned}$$

so consider the function

$$E(x) = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w),$$

and we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}x} &= -2\cdot\frac{\mathrm{d}w}{\mathrm{d}x}\cdot\left[(5\cosh x - 4\sinh x - 3)\cdot\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) - (w\cosh w + 2\sinh w)\right] \\ &= -2\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2(5\cosh x - 4\sinh x - 3) \\ &= -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\left[5\left(e^x + e^{-x}\right) - 4\left(e^x - e^{-x}\right) - 6\right] \\ &= -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\left(e^x + 9e^{-x} - 6\right) \\ &= -e^{-x}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2(e^x - 3)^2 \\ &\leq 0.\end{aligned}$$

Hence,

$$E(x) \leq E(0) = \left(\frac{1}{\sqrt{2}}\right)^2 + 2(0\sinh 0 + \cosh 0) = \frac{1}{2} + 2 = \frac{5}{2},$$

for $x \geq 0$.

Therefore,

$$\frac{5}{2} \geq \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w),$$

and hence

$$2(w\sinh w + \cosh w) \leq \frac{5}{2}$$

for $x \geq 0$ since squares are always non-negative.

Hence,

$$\cosh w \leq \frac{5}{4} - w\sinh w \leq \frac{5}{4}$$

for $x \geq 0$, the second inequality being true since $w\sinh w \geq 0$ since $\sinh w$ and w always take the same sign, as desired.