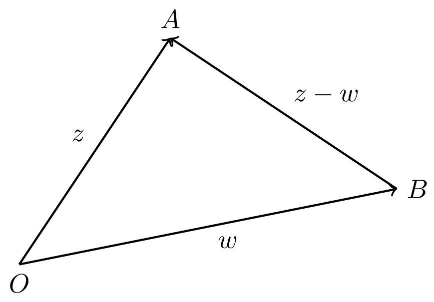
2013.3 Question 6



In the diagram, due to the triangular inequality, we must have $AB \leq OA + OB$, and hence $|z - w| \leq |z| + |w|$ as desired.

1. We have

$$\begin{aligned} \text{LHS} &= |z-w|^2 \\ &= (z-w)(z-w)^* \\ &= (z-w)(z^*-w^*) \\ &= zz^* + ww^* - zw^* - z^*w \\ &= |z|^2 + |w|^2 - (E - 2|zw|) \\ &= |z|^2 + 2|z||w| + |w|^2 - E \\ &= (|z|+|w|)^2 - E \\ &= \text{RHS}, \end{aligned}$$

   exactly as desired.

   Since $|z-w|$, $|z|$ and $|w|$ are all real, so must be $|z-w|^2$ and $(|z|+|w|)^2$, and so E must be real. Furthermore, we have

$$E = (|z|+|w|)^2 - |z-w|^2,$$

   and by the inequality $|z|+|w| \geq |z-w| \geq 0$, we can conclude

$$(|z|+|w|)^2 \geq |z-w|^2,$$

   and hence E must be non-negative.

2. We have

$$\begin{aligned} \text{LHS} &= |1 - zw^*|^2 \\ &= (1-zw^*)(1-zw^*)^* \\ &= (1-zw^*)(1-z^*w) \\ &= 1 - z^*w - zw^* + zwz^*w^* \\ &= 1 - (E - 2|zw|) + zw(zw)^* \\ &= 1 - (E - 2|zw|) + |zw|^2 \\ &= 1 + 2|zw| + |zw|^2 - E \\ &= (1+|zw|)^2 - E \\ &= \text{RHS}. \end{aligned}$$

If we square both sides of the desired inequality (since both sides are non-negative this is reversible), we have

$$\frac{|z-w|^2}{|1-zw^*|^2} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2},$$

which is equivalent to showing

$$\frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2}.$$

We introduce a lemma. If $a > c \geq 0$ and $a > b$, then

$$\frac{b-c}{a-c} \leq \frac{b}{a}.$$

The proof of this is as follows. We cross-multiply the inequality to give (since $a \geq a-c > 0$ this is reversible)

$$a(b-c) \leq b(a-c),$$

which is equivalent to

$$ac \geq bc,$$

and this must be true given $c \geq 0$ and $a > b$.

Now, since $|z| > 1$, $|w| > 1$, we have

$$(|z|-1)(|w|-1) = 1+|zw|-|z|-|w| > 0,$$

which means

$$1+|zw| > |z|+|w|,$$

and since both are non-negative we have

$$(1+|zw|)^2 > (|z|+|w|)^2.$$

Now, using this lemma, let $a = (1+|zw|)^2, b = (|z|+|w|)^2, c = E$. $a > b$ is as shown in above, and $c \geq 0$ is shown in part 1. $a > c$ since $a - c = |1-zw^*|^2 \geq 0$, and the equal sign holds if and only if $|zw^*| = |zw| = 1$, which must not hold if $|z| > 1$ and $|w| > 1$ since this gives $|zw| = |z||w| > 1$.

Therefore, we must have

$$\frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} \leq \frac{(|z|+|w|)^2}{(1+|zw|)^2},$$

which gives exactly what is desired.

This also holds for $|z| < 1$ and $|w| < 1$ since from this $(|z|-1)(|w|-1) > 0$ still holds, so $(1+|zw|)^2 > (|z|+|w|)^2$ remains true, and $|zw| = |z||w| < 1$ so $|zw| \neq 1$ remains true. The exact argument is still valid.