2013.3 Question 5

1. Since $q^n N = p^n$, we have $p^n \mid q^n N$, and hence $p \mid q^n N$.

   But since $\gcd(p, q) = 1$, we must have $p \mid q^{n-1} N$. Repeating this step we will get $p \mid N$.

   Let $N = pN_1$, we have $q^n pN_1 = p^n$, giving $q^n N_1 = p^{n-1}$. Repeating the same step will give $p \mid N_1$.

   Let $N_1 = pN_2$, we have $q^n pN_2 = p^{n-1}$, giving $q^n N_2 = p^{n-2}$. Repeating the same step will give $p \mid N_2$.

   We can repeat this until we reach $q^n N_{n-1} = p$ from which we can conclude $p \mid N_{n-1}$.

   So $N_{n-1} = kp$ for some $k \in \mathbb{N}$.

   But since $N_t = pN_{t+1}$, we can conclude that $N_1 = kp^{n-1}$ and hence

   $$N = pN_1 = kp^n$$

   as desired.

   Hence, we have $q^n kp^n = p^n$ which gives $q^n k = 1$. B33gut this means q^n and k must both be one since $q, k \in \mathbb{N}$. Hence, $q = 1$.

   Assume, for the sake of contradiction, that $\sqrt[n]{N}$ is a rational number that is not a positive integer. Let

   $$\sqrt[n]{N} = \frac{p}{q},$$

   where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, and $q \neq 1$ (this is to ensure it is not a positive integer).

   Hence, by rearrangement, we have

   $$q^n N = p^n,$$

   and from what we have proved we must have $q = 1$, which contradicts with $q \neq 1$.

   Hence, $\sqrt[n]{N}$ must either be a positive integer or must be irrational.

2. Since $a^a d^b = b^a c^b$, we know that $a^a \mid b^a c^b$. By the same reasoning as part 1, we know that $c^b = ka^a$ for some positive integer k_1.

   Hence, putting it back to the original equation, we have

   $$d^b = k_1 b^a,$$

   which implies $d^b \geq b^a$.

   Since $a^a d^b = b^a c^b$, we know that $c^b \mid a^a d^b$. By the same reasoning as part 1, we know that $a^a = k_2 c^b$ for some positive integer k_2.

   Hence, putting it back to the original equation, we have

   $$k_2 d^b = b^a,$$

   which implies $b^a \geq d^b$.

   This means $d^b = b^a$.

   If a prime $p \mid d$, then $p \mid d^b$, and hence $p \mid b^a$.

   Since $b^a = bb^{a-1}$, if p does not divide b, this means p and b must be co-prime (since p is a prime), then p must divide b^{a-1}, and repeating this argument eventually reaches p dividing $b^{a-(a-1)}$ which is a contradiction. So p must divide b.

   Let $d = p^m d'$, and we must have p not divide d'. Similarly, let $b = p^n b'$, and we must have p does not divide b'.

   Putting this back to $d^b = b^a$ shows

   $$(p^m d')^b = (p^n b')^a,$$

   and hence

   $$p^{mb} d'^b = p^{na} b'^a,$$

   and we must have p does not divide d'^b nor b'^a.

This means p^{mb} and p^{na} are exactly the highest powers of p that divide $d^b = b^a$, and hence

$$mb = na \iff b = \frac{na}{m}.$$

Since $p^n \mid b$, we must have $p^n \mid \frac{na}{m}$, and hence $p^n \mid na$. However, since a and b are co-prime, and p is a prime factor of b, then p must not divide a, and hence $p^n \mid n$. Hence, $p^n \leq n$.

Since $y^x > x$ for $y \geq 2$ and $x > 0$, and $p^n \leq n$, we must have $p < 2$ or $n \leq 0$. But since p is a prime, $p \geq 2$, so we must have $n \leq 0$ and hence $n = 0$.

This means that the highest power of the prime number p that divides b is always 0, and hence $b = 1$.

Let

$$r = \frac{p}{q},$$

where $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$.

We have

$$r^r = \frac{r}{s}$$

for $r, s \in \mathbb{N}$, $\gcd(r, s) = 1$.

We have

$$\begin{aligned}
\left(\frac{p}{q}\right)^{\frac{p}{q}} &= \frac{r}{s} \\
\left(\frac{p}{q}\right)^{p} &= \left(\frac{r}{s}\right)^{q} \\
p^p s^q &= q^p r^q.
\end{aligned}$$

Here, let $a = p, b = q, c = r$ and $d = s$. We must have $b = q = 1$, which contradicts with $q \neq 1$.

Therefore, $r = p \in \mathbb{N}$ is a positive integer.