2013.3 Question 4

We notice

$$(z-\exp(i\theta))(z-\exp(-i\theta)) = z^2-(\exp(i\theta)+\exp(-i\theta))z+1 = z^2-2z\cos\theta+1.$$

The $2n$-th roots of -1 are z_r, where $r=0,1,\dots,2n-1$,

$$z_r = \exp\left(i\left(\frac{\pi}{2n}+\frac{2r\pi}{2n}\right)\right) = \exp\left(i\pi\cdot\frac{1+2r}{2n}\right),$$

and hence

$$\begin{aligned}
z^{2n}+1 &= \prod_{r=0}^{2n-1}(z-z_r)\\
&= \left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=n}^{2n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\\
&= \left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2(2n-1-r)}{2n}\right)\right)\right]\\
&= \left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\right]\cdot\left[\prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{-1-2r}{2n}\right)\right)\right]\\
&= \prod_{r=0}^{n-1}\left(z-\exp\left(i\pi\cdot\frac{1+2r}{2n}\right)\right)\left(z-\exp\left(i\pi\cdot\frac{-1-2r}{2n}\right)\right)\\
&= \prod_{r=0}^{n-1}\left(z^2-2z\cos\left(\frac{2r+1}{2n}\pi\right)+1\right)\\
&= \prod_{r=1}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right).
\end{aligned}$$

1. Let $z=i$, since n is even, $z^{2n}=i^{2n}=(i^2)^n=(-1)^n=1$.

$$\begin{aligned}
2 &= z^{2n}+1\\
&= \prod_{r=1}^{n}\left(i^2-2i\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\\
&= \prod_{r=1}^{n}2i\cos\left(\frac{2r-1}{2n}\pi\right)\\
&= (2i)^n\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right)\\
&= 2^n(-1)^{\frac{n}{2}}\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right),
\end{aligned}$$

and therefore

$$\prod_{r=1}^{n}\cos\left(\frac{2r-1}{2n}\pi\right) = 2^{1-n}(-1)^{-\frac{n}{2}} = 2^{1-n}(-1)^{\frac{n}{2}}.$$

2. Notice that in the product where n is odd, let $k=\frac{n+1}{2}$, then the term of this product will be

$$\begin{aligned}
z^2-2z\cos\left(\frac{(2k-1)\pi}{2n}\right)+1 &= z^2-2z\cos\left(\frac{(n+1-1)\pi}{2n}\right)+1\\
&= z^2-2z\cos\frac{\pi}{2}+1\\
&= z^2+1.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
(z^2+1)\sum_{r=0}^{n-1}(-1)^r z^{2r} &= z^2+1 \\
&= \prod_{r=1}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)(z^2+1) \\
&\quad \prod_{r=\frac{n+3}{2}}^{n}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)(z^2+1) \\
&\quad \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2(n+1-r)-1}{2n}\pi\right)+1\right),
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sum_{r=0}^{n-1}(-1)^r z^{2r} &= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2-2z\cos\left(\frac{2(n+1-r)-1}{2n}\pi\right)+1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2-2z\cos\left(\frac{2n-2r+1}{2n}\pi\right)+1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}}\left(z^2-2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right)\left(z^2+2z\cos\left(\frac{2r-1}{2n}\pi\right)+1\right).
\end{aligned}
$$

Let $z=i$, we have

$$
\begin{aligned}
\text{LHS} &= \sum_{r=0}^{n-1}(-1)^r i^{2r} \\
&= \sum_{r=0}^{n-1}(-1)^r (i^2)^r \\
&= \sum_{r=0}^{n-1}(-1)^r (-1)^r \\
&= \sum_{r=0}^{n-1}[(-1)(-1)]^r \\
&= \sum_{r=0}^{n-1} 1 \\
&= n,
\end{aligned}
$$

and

$$\begin{aligned}
\text{RHS} &= \prod_{r=1}^{\frac{n-1}{2}} \left(i^2 - 2i\cos\left(\frac{2r-1}{2n}\pi\right) + 1\right)\left(i^2 + 2i\cos\left(\frac{2r-1}{2n}\pi\right) + 1\right) \\
&= \prod_{r=1}^{\frac{n-1}{2}} (-2i\cos\left(\frac{2r-1}{2n}\pi\right))(2i\cos\left(\frac{2r-1}{2n}\pi\right)) \\
&= \prod_{r=1}^{\frac{n-1}{2}} 4\cos^2\left(\frac{2r-1}{2n}\pi\right) \\
&= 2^{n-1}\prod_{r=1}^{\frac{n-1}{2}} \cos^2\left(\frac{2r-1}{2n}\pi\right).
\end{aligned}$$

This gives

$$\prod_{r=1}^{\frac{n-1}{2}} \cos^2\left(\frac{2r-1}{2n}\pi\right) = n2^{1-n},$$

exactly as desired.