2013.3 Question 3

Since $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4 = \mathbf{0}$, we must have

$$\begin{aligned} 0 &= \mathbf{0} \cdot \mathbf{0} \\ &= (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4) \cdot (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{4}_4) \\ &= \sum_{i=1}^{4} \mathbf{p}_i \cdot \mathbf{p}_i + 2 \sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathbf{p}_i \cdot \mathbf{p}_j. \end{aligned}$$

Since P_i are on the unit sphere, we must have $\mathbf{p}_i \cdot \mathbf{p}_i = 1$. By symmetry, for all $i \neq j$,

$$\mathbf{p}_i \cdot \mathbf{p}_j$$

must be some real constant, say k.

Hence,

$$0 = 4 \cdot 1 + 2 \cdot 6 \cdot k,$$

which solves to

$$k = -\frac{1}{3},$$

as desired.

1. We have

$$\begin{aligned} \sum_{i=1}^{4} (XP_i)^2 &= \sum_{i=1}^{4} (\mathbf{p}_i - \mathbf{x}) \cdot (\mathbf{p}_i - \mathbf{x}) \\ &= \sum_{i=1}^{4} (\mathbf{p_i} \cdot \mathbf{p_i} - 2\mathbf{x} \cdot \mathbf{p}_i + \mathbf{x} \cdot \mathbf{x}) \\ &= \sum_{i=1}^{4} \mathbf{p_i} \cdot \mathbf{p_i} - 2\mathbf{x} \cdot \sum_{i=1}^{4} + 4 \cdot \mathbf{x} \cdot \mathbf{x} \\ &= \sum_{i=1}^{4} 1 - 2\mathbf{x} \cdot \mathbf{0} + 4 \cdot 1 \\ &= 4 - 0 + 4 \\ &= 8. \end{aligned}$$

2. Since $P_1(0, 0, 1)$ and $P_2(a, 0, b)$, we must have

$$\mathbf{p}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{p}_2 = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix},$$

and hence

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = 0 \cdot a + 0 \cdot 0 + 1 \cdot b = b = -\frac{1}{3}.$$

We must have

$$|\mathbf{p}_2| = \sqrt{a^2 + 0^2 + b^2} = \sqrt{a^2 + b^2} = 1,$$

which means

$$a = \frac{2\sqrt{2}}{3},$$

as desired.

The z-component of $\mathbf{p}_3$ and $\mathbf{p}_4$ must also be $-\frac{1}{3}$, due to the dot product with $vectp_1$ being equal to the z-component must also be equal to $-\frac{1}{3}$.

Let

$$\mathbf{p}_3 = \begin{pmatrix} c \\ d \\ -\frac{1}{3} \end{pmatrix},$$

then from $\sum_{i=1}^{4} \mathbf{p}_i = \mathbf{0}$, we have

$$\mathbf{p}_4 = \begin{pmatrix} -c - \frac{2\sqrt{2}}{3} \\ -d \\ -\frac{1}{3} \end{pmatrix}.$$

Since $\mathbf{p}_3 \cdot \mathbf{p}_2 = -\frac{1}{3}$, we have

$$\frac{2\sqrt{2}}{3} \cdot c + 0 \cdot d + \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) = -\frac{1}{3},$$

and hence

$$\frac{2\sqrt{2}}{3} c = -\frac{4}{9},$$

which means

$$6\sqrt{2}c = -4,$$

and hence

$$c = -\frac{4}{6\sqrt{2}} = -\frac{\sqrt{2}}{3}.$$

Now, since $\mathbf{p}_3 \cdot \mathbf{p}_4 = -\frac{1}{3}$, we have

$$c \cdot \left(-c - \frac{2\sqrt{2}}{3}\right) + d \cdot (-d) + \left(-\frac{1}{3}\right) \cdot \left(-\frac{1}{3}\right) = -\frac{1}{3}.$$

Therefore,

$$\left(-\frac{\sqrt{2}}{3}\right) \cdot \left(-\frac{\sqrt{2}}{3}\right) - d^2 = -\frac{4}{9},$$

and hence

$$d^2 = \frac{2}{3},$$

giving

$$d = \pm\frac{\sqrt{2}}{\sqrt{3}}.$$

Hence,

$$P_3 \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right), P_4 \left(-\frac{\sqrt{2}}{\sqrt{3}}, \mp\frac{\sqrt{2}}{3}, -\frac{1}{3}\right).$$

3. We have

$$\begin{aligned} \sum_{i=1}^{4} (XP_i)^4 &= \sum_{i=1}^{4} \left[(\mathbf{p}_i - \mathbf{x}) \cdot (\mathbf{p}_i - \mathbf{x})\right]^2 \\ &= \sum_{i=1}^{4} (\mathbf{p}_i \cdot \mathbf{p}_i - 2\mathbf{x} \cdot \mathbf{p}_i + \mathbf{x} \cdot \mathbf{x})^2 \\ &= \sum_{i=1}^{4} (1 + 1 - 2\mathbf{x} \cdot \mathbf{p}_i)^2 \\ &= \sum_{i=1}^{4} (2 - 2\mathbf{x} \cdot \mathbf{p}_i)^2 \\ &= 4\sum_{i=1}^{4} (1 - \mathbf{x} \cdot \mathbf{p}_i)^2. \end{aligned}$$

Let $X(x, y, z)$. We have

$$\begin{aligned}
\sum_{i=1}^{4} (XP_i)^4 &= 4 \sum_{i=1}^{4} (1 - \mathbf{x} \cdot \mathbf{p}_i)^2 \\
&= 4 \left[\left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)^2 + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \frac{2\sqrt{2}}{3} \\ 0 \\ -\frac{1}{3} \end{pmatrix} \right)^2 \right. \\
&\quad \left. + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{3} \end{pmatrix} \right)^2 + \left(1 - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{3} \end{pmatrix} \right)^2 \right] \\
&= 4 \left[(1 - z)^2 + \left(1 - \frac{2\sqrt{2}}{3} x + \frac{1}{3} z \right)^2 \right. \\
&\quad \left. + \left(1 + \frac{\sqrt{2}}{3} x - \frac{\sqrt{2}}{\sqrt{3}} y + \frac{1}{3} z \right)^2 + \left(1 + \frac{\sqrt{2}}{3} x + \frac{\sqrt{2}}{\sqrt{3}} y + \frac{1}{3} z \right)^2 \right] \\
&= 4 \left(4 + \frac{4}{3} x^2 + \frac{4}{3} y^2 + \frac{4}{3} z^2 \right) \\
&= 4 \left[4 + \frac{4}{3} \right] \\
&= 4 \cdot \frac{16}{3} \\
&= \frac{64}{3}
\end{aligned}$$

is a constant, independent of the position of X.