2013.3 Question 2

We must have

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x} \cdot \frac{\arcsin x}{\sqrt{1-x^2}} \\
&= \frac{1}{1-x^2} \cdot \left(\frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} - \arcsin x \cdot (-2x) \cdot \left(\frac{1}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}} \right) \\
&= \frac{1}{1-x^2} \cdot \left(1 + x \cdot \frac{\arcsin x}{\sqrt{1-x^2}} \right) \\
&= \frac{1}{1-x^2} \cdot (1+xy),
\end{aligned}$$

which gives

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = (1+xy) - xy - 1 = 0$$

as desired.

Differentiating both sides of this equation w.r.t. x gives

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \cdot (1-x^2) - 2x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - y - x\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which combined gives

$$(1-x^2) \cdot \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - y = 0.$$

If we extend the definition of the differentiation operator to

$$\frac{\mathrm{d}^0y}{\mathrm{d}x^0} = y,$$

then this precisely proves the desired statement for the case $n = 0$ since $2n+3 = 3$ and $(n+1)^2 = 1$, and we will prove the desired statement for all non-negative integer n. The base case is shown as above.

Now, assume the given holds for some $n = k$ where k is a non-negative integer, i.e.

$$(1-x^2) \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+3)x \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (k+1)^2 \cdot \frac{\mathrm{d}^ky}{\mathrm{d}x^k} = 0,$$

we aim to show that the same holds for $n = k+1$.

Differentiating both sides with respect to x gives

$$(-2x) \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (1-x^2) \cdot \frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+3) \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (2k+3)x \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (k+1)^2 \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0,$$

which then simplifies to

$$(1-x^2) \cdot \frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+5)x \cdot \frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (k^2+4k+4) \cdot \frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0.$$

But notice that $n+2 = (k+1)+2 = k+3$, $n+1 = (k+1)+1 = k+2$, $(n+1)^2 = (k+2)^2 = k^2+4k+4$, $2n+3 = 2(k+1)+3 = 2k+5$, so this is exactly the statement when $n = k+1$, which finishes our inductive step.

Hence, by the Principle of Mathematical Induction, we can conclude that the original statement holds for any non-negative integer n, and hence for any positive integer n.

We have that

$$y|_{x=0} = \frac{\arcsin 0}{\sqrt{1-0^2}} = \frac{0}{1} = 0,$$

and evaluating the equation on the first derivative at $x = 0$ gives

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 1.$$

Evaluating the proven equation at $x = 0$ gives

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} = (n+1)^2 \left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0}.$$

Using this, we can conclude that

$$\left.\frac{\mathrm{d}^{2r}y}{\mathrm{d}x^{2r}}\right|_{x=0} = 0$$

for all $r \geq 0$ where r is an integer, since it is 0 when $n = 0$, and that

$$\left.\frac{\mathrm{d}^{2r+1}y}{\mathrm{d}x^{2r+1}}\right|_{x=0} = ((2r)!!)^2 = 2^{2r} \cdot (r!)^2$$

for all $r \geq 0$ where r is an integer, by mathematical induction.

Hence, the MacLaurin Series for $\frac{\arcsin x}{\sqrt{1-x^2}}$, must be

$$\begin{aligned}
\frac{\arcsin x}{\sqrt{1-x^2}} &= \sum_{k=0}^{\infty} \frac{\left.\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right|_{x=0}}{k!} \cdot x^k \\
&= \sum_{r=0}^{\infty} \frac{\left.\frac{\mathrm{d}^{2r} y}{\mathrm{d}x^{2r}}\right|_{x=0}}{(2r)!} \cdot x^{2r} + \sum_{r=0}^{\infty} \frac{\left.\frac{\mathrm{d}^{2r+1} y}{\mathrm{d}x^{2r+1}}\right|_{x=0}}{(2r+1)!} \cdot x^{2r+1} \\
&= 0 + \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1} \\
&= \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1}.
\end{aligned}$$

This means the general term for even powers of x is zero, and the general term for odd powers of x is

$$\frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot x^{2r+1}$$

where r is any non-negative integer.

The infinite sum can be expressed as

$$\sum_{r=0}^{\infty} \frac{(r!)^2}{(2r+1)!} = 2 \cdot \sum_{r=0}^{\infty} \frac{2^{2r} \cdot (r!)^2}{(2r+1)!} \cdot \left(\frac{1}{2}\right)^{2r+1},$$

which is precisely double the value of

$$\left[\frac{\arcsin x}{\sqrt{1-x^2}}\right]_{x=\frac{1}{2}} = \frac{\arcsin \frac{1}{2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\pi/6}{\sqrt{3}/2} = \frac{\pi}{3\sqrt{3}},$$

Hence, the sum evaluates to $\frac{2\pi}{3\sqrt{3}}$.