2013.3 Question 13

1. (a) Since $0 \leq X \leq 1$, we must have that

$$F(x) = \int_0^x f(t) \, \mathrm{d}t$$

for $0 \leq x \leq 1$. Hence, since $0 \leq f(t) \leq M$ for $0 \leq t \leq x \leq 1$, we have

$$0 = \int_0^x 0 \, \mathrm{d}t \leq F(x) \leq \int_0^x M \, \mathrm{d}t = Mx,$$

as desired.

(b) Since $0 \leq X \leq 1$, we must have $F(0) = 0$ and $F(1) = 1$. Let the desired integral be I, using integration by parts, we have

$$\begin{aligned} I &= \int_0^1 2g(x)F(x)f(x) \, \mathrm{d}x \\ &= \int_0^1 2g(x)F(x) \, \mathrm{d}F(x) \\ &= \left[2g(x)F(x)^2\right]_0^1 - 2\int_0^1 F(x) \, \mathrm{d}(g(x)F(x)) \\ &= 2g(1)F(1)^2 - 2g(0)F(0)^2 - 2\int_0^1 g'(x)F(x)^2 \, \mathrm{d}x - 2\int_0^1 g(x)F(x)f(x) \, \mathrm{d}x \\ &= 2g(1) - 2\int_0^1 g'(x)F(x)^2 \, \mathrm{d}x - I. \end{aligned}$$

This means

$$2I = 2g(1) - 2\int_0^1 g'(x)F(x)^2 \, \mathrm{d}x,$$

and hence

$$I = g(1) - \int_0^1 g'(x)F(x)^2 \, \mathrm{d}x.$$

2. (a) Since $0 \leq Y \leq 1$, we must have

$$\begin{aligned} \int_0^1 kF(y)f(y) \, \mathrm{d}y &= k\int_0^1 F(y) \, \mathrm{d}F(y) \\ &= k \cdot \frac{1}{2} \cdot \left[F(y)^2\right]_0^1 \\ &= k \cdot \frac{1}{2} \cdot \left[F(1)^2 - F(0)^2\right] \\ &= \frac{k}{2} \cdot (1^2 - 0^2) \\ &= \frac{k}{2} \\ &= 1, \end{aligned}$$

and hence $k = 2$.

(b) Notice that

$$\begin{aligned} \mathrm{E}\left(Y^n\right) &= \int_0^1 2y^n F(y)f(y) \, \mathrm{d}y \\ &\leq \int_0^1 2y^n My f(y) \, \mathrm{d}y \\ &= 2M\int_0^1 y^{n+1} f(y) \, \mathrm{d}y \\ &= 2M \, \mathrm{E}\left(X^{n+1}\right) \\ &= 2M\mu_{n+1}, \end{aligned}$$

and that

$$\begin{aligned}
\mathrm{E}\left(Y^n\right) &= \int_0^1 2y^n F(y) f(y) \, \mathrm{d}y \\
&= \left. y^n \right|_{y=1} - \int_0^1 (y^n)' F(y)^2 \, \mathrm{d}y \\
&= 1 - n \int_0^1 y^{n-1} F(y)^2 \, \mathrm{d}y \\
&\geq 1 - n \int_0^1 y^{n-1} M y F(y) \, \mathrm{d}y \\
&= 1 - Mn \int_0^1 y^n F(y) \, \mathrm{d}y \\
&= 1 - \frac{Mn}{n+1} \int_0^1 F(y) \, \mathrm{d}(y^{n+1}) \\
&= 1 - \frac{Mn}{n+1} \left(\left[F(y) y^{n+1} \right]_0^1 - \int_0^1 y^{n+1} \, \mathrm{d}F(y) \right) \\
&= 1 - \frac{Mn}{n+1} \left(F(1) \cdot 1^{n+1} - F(0) \cdot 0^{n+1} - \int_0^1 y^{n+1} f(y) \, \mathrm{d}y \right) \\
&= 1 - \frac{Mn}{n+1} \left(1 - \mathrm{E}\left(X^{n+1}\right) \right) \\
&= 1 - \frac{nM}{n+1} \mu_{n+1} - \frac{nM}{n+1},
\end{aligned}$$

as desired.

(c) Since we have for non-negative n,

$$1 + \frac{nM}{n+1} \mu_{n+1} - \frac{nM}{n+1} \leq 2M \mu_{n+1},$$

and hence for $n \geq 1$, we have

$$1 + \frac{(n-1)M}{n} \mu_n - \frac{(n-1)M}{n} \leq 2M \mu_n,$$

which multiplying both sides by n gives

$$n + (n-1)M\mu_n - (n-1)M \leq 2Mn\mu_n,$$

and rearranging gives

$$n - (n-1)M \leq M(n+1)\mu_n,$$

hence

$$\mu_n \geq \frac{n - (n-1)M}{M(n+1)} = \frac{n}{(n+1)M} - \frac{n-1}{n+1},$$

as desired.