2013.3 Question 12

1. Since $X_i \in \{0, 1\}$, we have $\mathrm{E}(X_i) = 0\,\mathrm{P}(X_i = 0) + 1\,\mathrm{P}(X_i = 1) = \mathrm{P}(X_i = 1)$.

   The total number of arrangements is
   $$\frac{n!}{a!b!}.$$

   To make $X_1 = 1$, we must have the first letter being A, and the rest can arrange to be whatever possible. Hence, the number of valid arrangements is
   $$\frac{(n-1)!}{(a-1)!b!}.$$

   Hence,
   $$\mathrm{E}(X_1) = v\,\mathrm{P}(X_1 = 1) = \frac{\frac{(n-1)!}{(a-1)!b!}}{\frac{n!}{a!b!}} = \frac{a}{n}.$$

   When $i \neq 1$, we must have the $i-1$th letter being B and the ith letter being A, and the rest can arrange to be whatever possible. Since $i > 1$, the $i-1$th letter will always exist. Hence, the number of valid arrangements is
   $$\frac{(n-2)!}{(a-1)!(b-1)!}$$

   Therefore,
   $$\mathrm{E}(X_i) = \mathrm{P}(X_i = 1) = \frac{\frac{(n-2)!}{(a-1)!(b-1)!}}{\frac{n!}{a!b!}} = \frac{ab}{n(n-1)}.$$

   Hence,
   $$\begin{aligned}\mathrm{E}(S) &= \mathrm{E}\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\ &= \frac{a}{n} + (n-1) \cdot \frac{ab}{n(n-1)} \\ &= \frac{a}{n} + \frac{ab}{n} \\ &= \frac{a(b+1)}{n}.\end{aligned}$$

2. (a) Notice that $X_1 X_j \in \{0, 1\}$, and $X_1 X_j = 1$ if and only if $X_1 = 1$ and $X_j = 1$. Hence,
   $$\mathrm{E}(X_1 X_j) = \mathrm{P}(X_1 = 1 \wedge X_j = 1).$$

   The arrangement for the event $X_1 = 1 \wedge X_j = 1$ must have the first letter A, the $j-1$-th letter B, and the j-th letter A. Since $j \geq 3$, we have $j - 1 \geq 2$ so will not repeat the requirement with the first letter. The rest can arrange whatever, so the number of valid arrangements is
   $$\frac{(n-3)!}{(a-2)!(b-1)!},$$

   and hence
   $$\mathrm{E}(X_1 X_j) = \mathrm{P}(X_1 = 1 \wedge X_j = 1) = \frac{\frac{(n-3)!}{(a-2)!(b-1)!}}{\frac{n!}{a!b!}} = \frac{a(a-1)b}{n(n-1)(n-2)},$$

   as desired.

(b) All terms in this sum satisfy $2 \leq i \leq n-2$ and $i+2 \leq j \leq n$. Notice that $X_i X_j \in \{0, 1\}$, and $X_i X_j = 1$ if and only if $X_i = 1$ and $X_j = 1$. Hence,

$$\mathrm{E}(X_i X_j) = \mathrm{P}(X_i = 1 \wedge X_j = 1).$$

The arrangement for the event $X_i = 1 \wedge X_j = 1$ must have the $i-1$-th letter B, i-th letter A, $j-1$-th letter B and j-th letter A. Since $j \geq i+2$, $j-1 \geq i+1 > i$, so the requirements do not repeat. Hence, the number of valid arrangements is

$$\frac{(n-4)!}{(a-2)!(b-2)!},$$

and hence

$$\mathrm{E}(X_i X_j) = \mathrm{P}(X_i = 1 \wedge X_j = 1) = \frac{\frac{(n-4)!}{(a-2)!(b-2)!}}{\frac{n!}{a!b!}} = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}.$$

The number of terms in this sum is

$$\begin{aligned}
\sum_{i=2}^{n-2} \sum_{j=i+2}^{n} 1 &= \sum_{i=2}^{n-2} (n - (i+2) + 1) \\
&= \sum_{i=2}^{n-2} (n - i - 1) \\
&= [(n-2) - 2 + 1](n-1) - \left[\frac{(n-2)(n-1)}{2} - 1\right] \\
&= (n-3)(n-1) - \left[\frac{n^2 - 3n}{2}\right] \\
&= (n-3)\left[(n-1) - \frac{n}{2}\right] \\
&= \frac{(n-3)(n-2)}{2}.
\end{aligned}$$

Hence, this sum evaluates to

$$\frac{(n-3)(n-2)}{2} \cdot \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} = \frac{a(a-1)b(b-1)}{2n(n-1)},$$

exactly as desired.

(c) To find $\mathrm{Var}(S)$, we would like to find $\mathrm{E}(S^2)$. Notice that

$$\begin{aligned}
\mathrm{E}(S^2) &= \mathrm{E}\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) \\
&= \mathrm{E}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} X_i X_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{E}(X_i X_j).
\end{aligned}$$

This sum can be further split up into these parts:

- Where $i = j$, the sum of $\mathrm{E}(X_i^2)$. But since X_i can only take 0 or 1, X_i^2 can only take 0 or 1, and we have

  $$\mathrm{P}(X_i = 0) = \mathrm{P}(X_i^2 = 0), \mathrm{P}(X_i = 1) = \mathrm{P}(X_i^2 = 1),$$

  and hence

  $$\mathrm{E}(X_i^2) = \mathrm{E}(X_i).$$

Hence, the sum can be evaluated as

$$\begin{aligned}\sum_{i=1}^{n} \mathrm{E}(X_i^2) &= \sum_{i=1}^{n} \mathrm{E}(X_i) \\ &= \mathrm{E}(X_1) + \sum_{i=2}^{n} \mathrm{E}(X_i) \\ &= \frac{a}{n} + (n-1) \cdot \frac{a(b+1)}{n(n-1)}.\end{aligned}$$

- Where $j = i \pm 1$. We can consider the case where $j = i + 1$ and double the result. For $X_i X_j = 1$, we must have $X_i = 1$ and $X_j = 1$, and hence the i-th letter must be A, and the $j-1$-th letter must be B. But this is impossible since $j = i + 1$, and a letter cannot be both A and B. And hence
$$2 \cdot \sum_{i=1}^{n-1} \mathrm{E}(X_i X_{i+1}) = 0.$$
- Where $j \geq i+2$ or $j \leq i-2$. We consider the case where $j \geq i+2$ and double the result. This is calculated in part a for the case $i = 1$, and part b for the case $i \geq 2$.

Hence,

$$\begin{aligned}\mathrm{E}(S^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}(X_i X_j) \\ &= \frac{a}{n} + (n-1) \cdot \frac{ab}{n(n-1)} + 2 \cdot \left[(n-2) \cdot \frac{a(a-1)b}{n(n-1)(n-2)} + \frac{a(a-1)b(b-1)}{2n(n-1)}\right] \\ &= \frac{a}{n} + \frac{ab}{n} + \frac{2a(a-1)b}{n(n-1)} + \frac{a(a-1)b(b-1)}{n(n-1)} \\ &= \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)} \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1}\right].\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{Var}(S) &= \mathrm{E}(S^2) - \mathrm{E}(S)^2 \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1}\right] - \left[\frac{a(b+1)}{n}\right]^2 \\ &= \frac{a(b+1)}{n}\left[1 + \frac{(a-1)b}{n-1} - \frac{a(b+1)}{n}\right] \\ &= \frac{a(b+1)}{n} \cdot \frac{n(n-1) + n(a-1)b - (n-1)a(b+1)}{n(n-1)} \\ &= \frac{a(b+1)}{n^2(n-1)}\left(n^2 - n + abn - nb - nab - na + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left(n^2 - n - nb - na + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left((a+b)^2 - (a+b) - (a+b)b - (a+b)a + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}\left(a^2 + 2ab + b^2 - a - b - ab - b^2 - a^2 - ab + ab + a\right) \\ &= \frac{a(b+1)}{n^2(n-1)}(ab - b) \\ &= \frac{a(b+1)}{n^2(n-1)} b(a-1) \\ &= \frac{a(a-1)b(b+1)}{n^2(n-1)}.\end{aligned}$$