2013.3 Question 1

Since $t = \tan \frac{1}{2}x$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2 \frac{1}{2}x = \frac{1}{2}(1 + \tan^2 \frac{1}{2}x) = \frac{1}{2}(1 + t^2).$$

By the tangent double-angle formula, we have

$$\tan x = \frac{2t}{1 - t^2},$$

and hence

$$\cot x = \frac{1 - t^2}{2t}.$$

Therefore,

$$\csc^2 x = 1 + \cot^2 x = 1 + \frac{(1 - t^2)^2}{(2t)^2} = \frac{(1 + t^2)^2}{(2t)^2},$$

which means

$$\sin^2 x = \frac{(2t)^2}{(1 + t^2)^2},$$

and hence

$$|\sin x| = \frac{2t}{1 + t^2}.$$

What remains is to consider the sign. Notice that $t \geq 0$ if and only if

$$\frac{x}{2} \in \bigcup_{k \in \mathbb{Z}} \left[k\pi, k\pi + \frac{\pi}{2}\right),$$

which is

$$x \in \bigcup_{k \in \mathbb{Z}} [2k\pi, 2k\pi + \pi),$$

but this is also precisely if and only if $\sin x \geq 0$.

This means $\sin x$ must take the same sign as t, and hence

$$\sin x = \frac{2t}{1 + t^2}.$$

Using this substitution, we have when $x = 0, t = 0$ and when $x = \frac{1}{2}\pi, t = 1$, and also

$$\mathrm{d}x = \frac{2\,\mathrm{d}t}{1 + t^2}.$$

This means

$$\begin{aligned}
I &= \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{1 + a\sin x} \\
&= \int_0^1 \frac{\frac{2\,\mathrm{d}t}{1+t^2}}{1 + a \cdot \frac{2t}{1+t^2}} \\
&= \int_0^1 \frac{2\,\mathrm{d}t}{1 + 2at + t^2} \\
&= \int_0^1 \frac{2\,\mathrm{d}t}{(t + a)^2 + (1 - a^2)} \\
&= \frac{2}{1 - a^2} \int_0^1 \frac{\mathrm{d}t}{\left(\frac{t+a}{\sqrt{1-a^2}}\right)^2 + 1} \\
&= \frac{2}{1 - a^2} \cdot \sqrt{1 - a^2} \cdot \left[\arctan\left(\frac{t + a}{\sqrt{1 - a^2}}\right)\right]_0^1 \\
&= \frac{2}{\sqrt{1 - a^2}} \cdot \left[\arctan\left(\frac{1 + a}{\sqrt{1 - a^2}}\right) - \arctan\left(\frac{a}{\sqrt{1 - a^2}}\right)\right].
\end{aligned}$$

But notice that

$$\begin{aligned}\arctan\left(\frac{1+a}{\sqrt{1-a^2}}\right)-\arctan\left(\frac{a}{\sqrt{1-a^2}}\right)&=\arctan\left(\frac{\frac{1+a}{\sqrt{1-a^2}}-\frac{a}{\sqrt{1-a^2}}}{1+\frac{1+a}{\sqrt{1-a^2}}\cdot\frac{a}{\sqrt{1-a^2}}}\right)\\&=\arctan\left(\frac{\frac{1}{\sqrt{1-a^2}}}{1+\frac{a+a^2}{1-a^2}}\right)\\&=\arctan\left(\frac{\sqrt{1-a^2}}{(1-a^2)+(a+a^2)}\right)\\&=\arctan\left(\frac{\sqrt{1-a}\cdot\sqrt{1+a}}{1+a}\right)\\&=\arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right),\end{aligned}$$

and hence

$$I=\frac{2}{\sqrt{1-a^2}}\arctan\left(\frac{\sqrt{1-a}}{\sqrt{1+a}}\right),$$

as desired.

We have

$$\begin{aligned}I_{n+1}+2I_n&=\int_0^{\frac{1}{2}\pi}\frac{\sin^{n+1}x+2\sin^n x}{2+\sin x}\,\mathrm{d}x\\&=\int_0^{\frac{1}{2}\pi}\sin^n x\,\mathrm{d}x.\end{aligned}$$

Therefore, we have

$$\begin{aligned}I_3+2I_2&=\int_0^{\frac{1}{2}\pi}\sin^2 x\,\mathrm{d}x\\&=\int_0^{\frac{1}{2}\pi}\frac{1-\cos 2x}{2}\,\mathrm{d}x\\&=\left[\frac{1}{2}\cdot x-\frac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\\&=\left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{4}\sin\pi\right)-\left(\frac{1}{4}\sin 0-\frac{1}{2}\cdot 0\right)\\&=\frac{\pi}{4},\end{aligned}$$

$$\begin{aligned}I_2+2I_1&=\int_0^{\frac{1}{2}\pi}\sin x\,\mathrm{d}x\\&=[-\cos x]_0^{\frac{1}{2}\pi}\\&=\left(-\cos\frac{1}{2}\pi\right)-(-\cos 0)\\&=(0)-(-1)\\&=1,\end{aligned}$$

and

$$\begin{aligned}I_1+2I_0&=\int_0^{\frac{1}{2}\pi}\sin^0 x\,\mathrm{d}x\\&=[x]_0^{\frac{1}{2}\pi}\\&=\frac{1}{2}\pi.\end{aligned}$$

Also, notice that

$$\begin{aligned}
I_0 &= \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{2+\sin x} \\
&= \frac{1}{2} \int_0^{\frac{1}{2}\pi} \frac{\mathrm{d}x}{1+\frac{1}{2}\sin x} \\
&= \frac{1}{2} \cdot \frac{2}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \cdot \arctan \frac{\sqrt{1-\frac{1}{2}}}{\sqrt{1+\frac{1}{2}}} \\
&= \frac{1}{2} \cdot \frac{4}{\sqrt{3}} \cdot \arctan \frac{1}{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} \\
&= \frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I_3 &= \frac{\pi}{4} - 2I_2 \\
&= \frac{\pi}{4} - 2 \cdot (1 - 2I_1) \\
&= \frac{\pi}{4} - 2 + 4I_1 \\
&= \frac{\pi}{4} - 2 + 4\left(\frac{1}{2}\pi - 2I_0\right) \\
&= \frac{\pi}{4} - 2 + 2\pi - 8I_0 \\
&= \frac{9\pi}{4} - 2 - \frac{8\pi}{3\sqrt{3}} \\
&= \left(\frac{9}{4} - \frac{8}{3\sqrt{3}}\right)\pi - 2.
\end{aligned}$$