2012.3 Question 8

1. We aim to show that for all $n \geq 0$,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_{n+2} F_{n+5} - F_{n+3} F_{n+4}.$$

Notice that

$$\begin{aligned}
\text{RHS} &= F_{n+2} F_{n+5} - F_{n+3} F_{n+4} \\
&= F_{n+2}(F_{n+3} + F_{n+4}) - F_{n+3}(F_{n+2} + F_{n+3}) \\
&= F_{n+2} F_{n+4} - F_{n+3} F_{n+3} \\
&= F_{n+2}(F_{n+2} + F_{n+3}) - F_{n+3}(F_{n+1} + F_{n+2}) \\
&= F_{n+2} F_{n+2} - F_{n+3} F_{n+1} \\
&= F_{n+2}(F_{n+3} - F_{n+1}) - F_{n+3}(F_{n+2} - F_n) \\
&= F_n F_{n+3} - F_{n+1} F_{n+2} \\
&= \text{LHS}
\end{aligned}$$

and set $n = 0$ shows exactly what is desired.

2. By the lemma in the previous part, the problem reduces to two cases are when n is odd and when n is even.

   - When n is even,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_0 F_3 - F_1 F_2 = 0 \cdot 2 - 1 \cdot 1 = -1.$$

   - When n is odd,

$$F_n F_{n+3} - F_{n+1} F_{n+2} = F_1 F_4 - F_2 F_3 = 1 \cdot 3 - 1 \cdot 2 = 1.$$

3. Using the tangent formula for sum of angles, we have

$$\begin{aligned}
\arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right) &= \arctan\left(\frac{\frac{1}{F_{2r+1}} + \frac{1}{F_{2r+2}}}{1 - \frac{1}{F_{2r+1}} \cdot \frac{1}{F_{2r+2}}}\right) \\
&= \arctan\left(\frac{F_{2r+1} + F_{2r+2}}{F_{2r+1} F_{2r+2} - 1}\right) \\
&= \arctan\left(\frac{F_{2r+3}}{F_{2r+1} F_{2r+2} + (F_{2r} F_{2r+3} - F_{2r+1} F_{2r+2})}\right) \\
&= \arctan\left(\frac{F_{2r+3}}{F_{2r} F_{2r+3}}\right) \\
&= \arctan\left(\frac{1}{F_{2r}}\right),
\end{aligned}$$

as desired.

Hence, we have

$$\arctan\left(\frac{1}{F_{2r+1}}\right) = \arctan\left(\frac{1}{F_{2r}}\right) - \arctan\left(\frac{1}{F_{2r+2}}\right),$$

and therefore

$$\begin{aligned}\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right) &= \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) - \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+2}}\right) \\ &= \sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) - \sum_{r=2}^{\infty} \arctan\left(\frac{1}{F_{2r}}\right) \\ &= \arctan\left(\frac{1}{F_{2\cdot 1}}\right) \\ &= \arctan\left(\frac{1}{F_{2}}\right) \\ &= \arctan(1) \\ &= \frac{\pi}{4}.\end{aligned}$$